APPLICATION OF NUMERICAL INVERSION OF THE LAPLACE TRANSFORM TO THE INVERSE PROBLEM IN TRANSIENT HEAT CONDUCTION

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THE LAPLACE TRANSFORM TO THE INVERSE
PROBLEM IN TRANSIENT HEAT CONDUCTION

by

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ABSTRACT

The direct problem in transient heat conduction requires the determination of conditions at an interior location when conditions are known at the boundaries of a solid. Conversely, the inverse problem requires the determination of conditions at the boundaries of a solid when conditions are known at an interior location. Consequently special methods are required in the solution of the inverse problem. A new method, numerical inversion of the Laplace transform, is used to solve this complex problem. Application of this numerical technique to the semi-infinite [redacted] cylinder, and sphere is made, and the accuracy of solution is discussed. This method of solution provides the engineer with a simple, powerful tool that can be used in the determination of heat transfer phenomena in a solid.

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LIST OF SYMBOLS

t	Time variable
u	Temperature function
L	Laplacian Operator (a linear operator)
$f(t)$	Any transformable function of time
$F(s)$	Laplace transform of f(t)
$w(x)$	Weighting function
x_i	Real and distinct roots of the shifted Legendre polynomials
$\pi(x)$	A general orthogonal polynomial defined by equation (5)
$P_N(r)$	Legendre polynomial of r and degree N
A_i	Coefficients (or weights) of the Legendre polynomials
C_i	Coefficients of the polynomials $\pi(x)$ defined in equation (11)
$\prod$	Product Symbol
b_k	Coefficients of the Legendre polynomials, defined in equation (22), or coefficients of the Laguerre polynomials defined in equation (67)
$P_N^*(x)$	Shifted Legendre polynomial
r_i	Real and distinct roots of the Legendre polynomials
A_i^*	Coefficients (or weights) of the shifted Legendre polynomials
log	Natural logarithm of the function indicated
$\alpha(x)$	Transformed temperature function, defined in equation (13)
y_i	Variable equal to $A_i^* \alpha(x_i)$
a_k	Variable equal to $F(k+1)$

q_{kj}	The explicit inverse matrix
$P_N^{*'}(x)$	Derivative of the shifted Legendre polynomial
$\hat{f}(x)$	An even function equal to $f(-\frac{1}{\gamma}\log x)$
γ	Scale factor
K	Constant equal to $\frac{k}{Cp(\text{density})}$
k	Thermal conductivity

I. INTRODUCTION

Traditionally, heat-transfer problems are concerned with the determination of conditions at an interior location when conditions are known at the boundaries of a solid. Such problems are known as direct problems, and their solutions are obtained through straightforward classical techniques [5].* In many applications (i.e., re-entry thermodynamics, rocket motors, quenching), it is desired to determine the conditions at the boundaries of a solid when conditions are known at an interior location. Problems of this type are known as inverse problems and require special methods of solution.

Many solutions to the inverse problem of heat conduction exist in the literature. Stolz [19] uses numerical inversion of a specific direct problem to solve for the heat flux at the surface of a solid. He notes that there is appreciable damping or lag in time of the heat flux from the interior location to the surface. This damping resulted in oscillations of the approximate solution with resultant errors. Burggraf [4] obtains an "exact" solution to the inverse problem in heat conduction with internal heat generation. However Beck [4] in commenting on this solution notes that this "exact" method must utilize inexact measured temperature data for which accurate time derivatives (necessary for this "exact" solution) are very difficult to obtain. In commenting on the inaccuracies involved in using measured temperature data, Frank [12] suggests using a least squares fit to the data available in order to obtain an essentially smooth temperature function for use in the solution of the inverse problem. Sparrow, et. al. [18] use a modification of Laplace transform inversion

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*Numbers in brackets refer to references in the bibliography.

by applying a convergence factor to the temperature function in the s plane. Deverall and Channapragada [11] utilize a convergence factor to obtain a solution for the heat flux in an inverse problem.

It is intended here to present a simple straightforward method of obtaining solutions to inverse problems of practical interest. The numerical technique involved in inverting the Laplace transform (used in solving the inverse problems) was developed by Bellman and Kalaba [2 and 3]. This method, based on orthogonal function theory and Gaussian quadrature, is developed and tested for accuracy (Tables I and II). This numerical technique is applied to the semi-infinite solid, the "long" cylinder and the sphere. The accuracy of solution is compared graphically with the known exact solutions. The FORTRAN IV programs used in the solutions are listed in Appendix A for use by the reader.

II. NUMERICAL TECHNIQUE

INTRODUCTION

The numerical technique involved was developed by Bellman and Kalaba [2] and involves the application of orthogonal function theory to the inversion of the Laplace Transform as opposed to using the Convolution Theorem which involves integration in the complex plane.

It is important to note that the method developed in this section can be applied to other problems in physics, biology, economics, chemotherapy and radiation [2].

THE LAPLACE TRANSFORM

We will first consider some of the properties of the Laplace transform and limitations involved in applying the inverse.

Let f(t) be any sectionally continuous function defined for $t \geq 0$. Then the Laplace transform of f(t) is defined as

$$F(s) = L\{f(t)\} = \int_0^\infty e^{-st} f(t)dt \qquad (1)$$

where the real part of s is large enough to make the integral (1) convergent.

The Laplacian Operator L is a linear operator and possesses many desirable properties, mathematically and computationally. Some of these properties are illustrated in the following theorems which are included here for completeness:

<u>THEOREM</u> <u>I</u>: If $L\{f_1(t)\} = F_1(s)$ and $L\{f_2(t)\} = F_2(s)$ then $L\{f_1(t) + f_2(t)\} = F_1(s) + F_2(s)$.

<u>THEOREM</u> <u>II</u>: If $L\{f(t)\} = F(s)$ then $L\{\frac{\partial f(t)}{\partial t}\} = sF(s) - f(0)$.

THEOREM III: If $L\{f(t)\} = F(s)$

$$\text{then } L\left\{\frac{\partial^N f(t)}{\partial x^N}\right\} = \frac{d^N}{dx^N}\left[F(s)\right].$$

THEOREM IV: If $L\{f(t)\} = F(s)$

$$\text{then } L\left\{\int_0^t f(\bar{t})d\bar{t}\right\} = \frac{1}{s} F(s).$$

THEOREM V: If $L\{f(t)\} = F(s)$

$$\text{then } L\{f(at)\} = \frac{1}{a} F\left(\frac{s}{a}\right).$$

THEOREM VI: Initial Value Theorem.

If $L\{f(t)\} = F(s)$

then

$$\underset{s\to\infty}{\text{Limit}}\ sF(s) = \underset{t\to\infty}{\text{Limit}}\ f(t).$$

THEOREM VII: Final Value Theorem.

If $L\{f(t)\} = F(s)$

then

$$\underset{s\to\infty}{\text{Limit}}\ sF(s) = \underset{t\to\infty}{\text{Limit}}\ f(t).$$

THEOREM VIII: Superposition Theorem.

If $L\{f(t)\} = F(s)$ and $L\{g(t)\} = G(s)$

then

$$L\left\{\int_0^t f(\bar{t})g(t-\bar{t})d\bar{t}\right\} = L\left\{\int_0^t g(\bar{t})f(t-\bar{t})d\bar{t}\right\}$$

$$= F(s)G(s)$$

INSTABILITY OF THE INVERSE OF THE LAPLACE TRANSFORM

The instability of the inverse of the Laplace transform can be illustrated by considering the function

$$f(t) = \sin \omega t.$$

The Laplace transform of f(t) is then

$$L\{\sin \omega t\} = \frac{\omega}{s^2+\omega^2} \qquad (2)$$

Now it is known that sin ωt is bounded in amplitude by (1,-1); but as ω is increased, sin ωt oscillates correspondingly more rapidly. However the Laplace transform of sin ωt approaches zero as ω is increased.

$$\operatorname*{Limit}_{\omega\to\infty} \frac{\omega}{s^2+\omega^2} = 0$$

In other words small changes in s can cause arbitrarily large changes in t with respect to the period of oscillation. Consequently we cannot sift or filter out rapid oscillations by using values of F(s). Such difficulties as mentioned above can be avoided only if f(t) is an essentially smooth function, which is an implicit assumption in the following development.

NUMERICAL QUADRATURE

There are several methods that can be used to approximate a definite integral in terms of a finite sum. The method employed here is based on Gaussian Quadrature and is used because of convergence properties and exactness of solution [15]. Consider the integral approximation

$$\int_a^b w(x)f(x)dx = \sum_{i=1}^{M} A_i f(x_i). \qquad (3)$$

By minimizing the function

$$\left[w(x)f(x) - \sum_{i=1}^{M} A_i f(x_i)\right]^2, \qquad (4)$$

we can assume that the summation involved in equation (3) is exact for a prescribed accuracy, ε, for any polynomial of degree less than or equal to M. The x_i's and the A_i's are to be determined so that equation (3) is exact for a polynomial f(x) of high degree. Here the x_i's are called the roots (or nodes), the A_i's are called the coefficients (or weights), and w(x) is called the weighting function of the function f(x).

With this background let us examine the numerical approximation to an integral which will be used later to invert the Laplace transform. Let

$$\pi(x) = \prod_{i=1}^{M} (x-x_i) \; , \tag{5}$$

and let $P_N(x)$ be a polynomial of degree N. Then we have

$$P_N(x) = \pi(x)Q_{N-M}(x) + P_{M-1}(x). \tag{6}$$

Since $\quad P_N(x_i) = P_{M-1}(x_i) \qquad\qquad i = 1,2,\ldots,M$

then $$\int_a^b w(x)P_N(x)dx = \int_a^b w(x)P_{M-1}(x)dx + \int_a^b w(x)\pi(x)Q_{N-M}(x)dx. \tag{7}$$

Now let us choose the x_i's such that

$$\int_a^b w(x)\pi(x)Q_{N-M}(x)dx = 0, \tag{8}$$

for all polynomials $Q_{N-M}(x)$. Then from equation (3) we have

$$\int_a^b w(x)P_N(x)dx = \sum_{i=1}^{M} A_i P_N(x_i). \tag{9}$$

If equation (8) is to be valid for all polynomials Q_{N-M} for $N \geq M$, it must be valid for

$$\int_a^b w(x)\pi(x)x^{\kappa}dx = 0, \qquad\qquad \kappa = 1,2,\ldots,N-M \tag{10}$$

Let us now expand $\pi(x)$ into a polynomial

$$\pi(x) = \sum_{i=o}^{M} C_i x^i, \tag{11}$$

so that equation (10) becomes

$$\int_a^b w(x)\left(\sum_{i=o}^{M} C_i x^i\right) x^K dx = 0, \qquad i = 0,1,2,\ldots,N\text{-}M \tag{12}$$

which is a system of M equations in the c_i's if $N = 2M-1$.

The system of equations in (12) has a unique solution <u>if and only if</u> the matrix of coefficients is nonsingular.

The determinant of this matrix is called a Vandermonde determinant and can be easily evaluated, since $w(x) \geq 0$ in the inverval (a,b), as

$$\begin{vmatrix} 1 & x_o & \cdots & x_o^N \\ 1 & x_1 & \cdots & x_1^N \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ 1 & x_N & \cdots & x_N^N \end{vmatrix} = \prod_{i>j} (x_i - x_j). \tag{13}$$

Since the x_i's are real and distinct points (roots), the determinant in (13) does not vanish and the b_i's can be uniquely determined.

It can be shown that the zeros of $\pi(x)$, determined by its coefficients (c_i), are real and in the interval (a,b). It can also be shown that $Q_{N-M}(x)$ is a polynomial of degrees M-1, where $N = 2M-1$.

Now let $\pi_i(x)$ and $\pi_j(x)$ designate the functions $\pi_M(x)$ determined by their coefficients for $M = i$ and $M = j$ respectively. Then

$$\int_a^b w(x)\pi_i(x)\pi_j(x)dx = 0 \qquad i \neq j \tag{14}$$

If $i < j$, then $\pi_i(x)$ is a polynomial of degree less than j; and the result follows from equation (8), which has been verified. From equation (14), $\pi_i(x)$ and $\pi_j(x)$ are said to be orthogonal on the interval (a,b) with respect to the weight function w(x).

For each value of M, the x_i's in equation (3) are taken as the zeros of $\pi_M(x)$ as determined by its coefficients. The A_i's in equation (3) can then be determined by the method of undetermined coefficients. It can be shown that all of the A_i's are positive.

Therefore for a given weight function w(x) and interval of integration (a,b), the zeros of the polynomials $\pi_M(x)$, as determined by their coefficients (c_i), need be computed <u>only once</u>. Likewise the corresponding A_i's need be computed <u>only once</u>.

GAUSS-LEGENDRE QUADRATURE

Since we have been dealing with any orthogonal function $\pi_M(x)$, let us indicate the specific orthogonal functions used in the following development. The Legendre polynomials are the unique Gauss quadrature polynomials [15] in the interval (-1,1) (or any finite interval) with weighting function

$$w(r) \equiv 1 \tag{15}$$

Let us denote the Legendre polynomials, in accordance with common practice, by the notation $P_N(r)$ where

$$P_N(r) = \pi(r)\,\frac{(2N)!}{2^N(N!)^2}\,, \tag{16}$$

and the interval of expansion is (-1,1). These polynomials are defined by the Rodrigue's formula

$$P_N(r) = \frac{1}{2^N N!}\,\frac{d^N}{dr^N}\left(r^2-1\right)^N \;. \tag{17}$$

From equation (14) it follows that

$$\int_{-1}^{1} P_N(r)P_M(r)\,dr = 0 \qquad\qquad M \neq N\,. \tag{18}$$

The recursion relations for the Legendre polynomials are then

$$P_{N+1}(r) = \left(\frac{2N-1}{N+1}\right)r\,P_N(r) - \left(\frac{N}{N+1}\right)P_{N-1}(r), \tag{19}$$

and

$$P_{N+1}(r) = r\,P_N(r) + \left(\frac{r^2-1}{N+1}\right)P_N'(r). \tag{20}$$

Consequently the least squares polynomial approximation (Gaussian Quadrature) to f(x) over (-1,1) with respect to constant weighting function is

$$f(r) = \sum_{K=0}^{N} b_K P_K(r), \tag{21}$$

where $$b_K = \frac{2K+1}{2} \int_{-1}^{1} f(r) P_K(r) dr. \tag{22}$$

Therefore the coefficients (or weights) A_i are

$$A_i = \int_{-1}^{1} \frac{P_N(r)}{(r-r_i)\, P_N{}^1(r)}\, dr\ , \qquad (23$$

where r_i's are the roots of the Legendre polynomials $P_N(r)$.

SHIFTED LEGENDRE POLYNOMIALS

We have now examined a specific orthogonal polynomial, the Legendre polynomial. However, it is most convenient in the following development to consider the shifted Legendre polynomials. The reason for employing these polynomials is twofold. First, the interval of expansion (0,1) of the shifted Legendre polynomials corresponds to that involved in obtaining an explicit inverse matrix. Second, the coefficients of the shifted Legendre polynomials are all integers, which is an important property computationally.

Since the roots and weights have been tabulated for the interval (-1,1) [19], it is a simple matter to obtain the roots and weights of the shifted Legendre polynomials in the interval (0,1).

Let

$$P_N{}^*(x) = P_N(1-2r) \tag{24}$$

where $P_N{}^*(x)$ are the shifted Legendre polynomials. Consequently the roots of the shifted Legendre polynomials are

$$x_i = \frac{1+r_i}{2}\ , \tag{25}$$

and the weights are

$$A_i{}^* = \frac{A_i}{2}\ . \tag{26}$$

NUMERICAL INVERSION OF THE LAPLACE TRANSFORM

We have now laid a proper foundation for obtaining an explicit inverse matrix to be used in inverting the Laplace transform. Let us consider the approximate inversion of the Laplace transform through numerical techniques.

Let

$$F(s) = \int_0^\infty e^{-st} u(t)dt. \qquad (27)$$

Now we make a change of the variable of integration to

$$x = e^{-t} \qquad (28)$$

therefore

$$t = -\log x \qquad (29)$$

Substituting equation (29) into equation (27) we obtain

$$\int_0^1 x^{s-1} u(-\log x)dx = F(s). \qquad (30)$$

Now let $\alpha(x) = u(-\log x)$ (31)

and then equation (30) becomes

$$\int_0^1 x^{s-1} \alpha(x)dx = F(s). \qquad (32)$$

If we make $g(x) = \alpha(x)x_i^{s-1}$, then using the quadrature formula

$$\int_0^1 g(x)dx = \sum_{i=1}^N A_i^* g(x_i), \qquad (33)$$

we obtain

$$\sum_{i=1}^N A_i^* x_i^{s-1} \alpha(x_i) = F(s). \qquad (34)$$

Letting $s = 1,2,\ldots,N$, we obtain a linear system of N equations in the N unknowns, $\alpha(x_i)$, $i = 1,2,\ldots,N$,

$$\sum_{i=1}^N A_i^* x^\kappa \alpha(x_i) = F(\kappa+1) \qquad \kappa = 0,1,\ldots,N-1 \qquad (35)$$

It is our intention here to obtain

$$\alpha(x_i) = \sum_{\kappa=0}^{N-1} a_{i\kappa} F(\kappa+1), \qquad i = 1,2,\ldots,N, \qquad (36)$$

where the $a_{i\kappa}$'s are constants independent of $F(\kappa+1)$, determined once and for all for each N, and tabulated.

Let us now make another change of variables such that

$$y_i = A_i^* \alpha(x_i), \tag{37}$$

and

$$a_\kappa = F(\kappa+1). \tag{38}$$

Equation (35) then becomes

$$\sum_{i=1}^{N} x_i^\kappa y_i = a_\kappa , \qquad \kappa = 0,1,\ldots,N-1. \tag{39}$$

Multiplying the κ^{th} equation by a parameter q_κ, and adding the corresponding terms in all the equations, we get

$$\sum_{i=1}^{N} y_i \left(\sum_{\kappa=o}^{N-1} q_\kappa x_i^\kappa \right) = \sum_{\kappa=o}^{N-1} a_\kappa q_\kappa . \tag{40}$$

Now setting

$$f(x) = \sum_{\kappa=o}^{N-1} q_\kappa x^\kappa , \tag{41}$$

we have

$$\sum_{i=1}^{N} y_i f(x_i) = \sum_{\kappa=o}^{N-1} a_\kappa q_\kappa , \tag{42}$$

where $f(x)$ is a polynomial of degree N-1, to be chosen in some convenient manner.

We want to determine the y_j's, and thus we require that $f(x) \equiv f_j(x)$ such that

$$f_j(x_i) = \delta_{ij} = \begin{cases} 0 & i \neq j , \\ 1 & i = j \end{cases} \tag{43}$$

an orthogonality condition. If

$$f_j(x) = \sum_{\kappa=o}^{N-1} q_{\kappa j} \, x^\kappa , \tag{44}$$

equation (42) reduces to

$$y_j = \sum_{\kappa=o}^{N-1} a_\kappa q_{\kappa j} , \tag{45}$$

where $q_{\kappa j}$, $\kappa = 0,1,\ldots,N-1$, are determined by the orthogonality condition in equation (43).

Since the interval of integration involved in the preceding development is (0,1), we will use the shifted Legendre polynomials in approximating the integral in equation (32). Because the x_i's are

the zeros of the shifted Legendre polynomial $P_N^*(x)$, we can use the Lagrange interpolation formula

$$f_j(x) = \frac{P_N^*(x)}{(x-x_j)\, P_N^{*\prime}(x_j)} . \tag{46}$$

The desired q_{Kj}'s in equation (45) are the coefficients in this polynomial of degree N-1. Repeating this procedure for each j yields the desired inverse matrix (q_{Kj}).

INTERLACING OF THE ROOTS

It should be noted that the roots (x_i) for different N interlace (Appendix B). Consequently the values of time

$$t_i = -\log x_i \tag{47}$$

intertwine, and the results for one N may be combined with those of another N to obtain both a check on the results and an enlarged set of t values for extrapolation purposes.

CHANGE OF TIME SCALE

The method of inversion described for the Laplace transform provides approximate values for the function u(t) at discrete values of t. For general N, we obtain $u(t_i)$ at the points $t_i = -\log x_i$ where x_i are the roots (or zeros) of the shifted Legendre polynomials, $P_N^*(x)$.

It is obvious that the inverse transform is thus determined for a very small range of t values. Since increasing N does not significantly increase the range of t, we must resort to another method of time scaling. Recall that if

$$F(s) = L\{u(t)\} = \int_0^\infty e^{-st} u(t)dt, \tag{48}$$

then from Theorem V, it follows that

$$L\{u(at)\} = \frac{1}{a} F\left(\frac{s}{a}\right). \tag{49}$$

Therefore using the approximate relation

$$\sum_{i=1}^{N} A_i x_i^{s-1} u(-a \log x_i) \simeq \frac{1}{a} F\left(\frac{s}{a}\right) , \tag{50}$$

we can determine u(t) for values of $t_i = -a \log x_i$.

ALTERNATE METHODS

Although the method of inverting the Laplace transform presented involves the use of the shifted Legendre polynomials, there are several other methods that can be used in obtaining an explicit inverse matrix. The first is the Method of Papoulis which utilizes the Legendre polynomials and the second is based on a different set of orthogonal polynomials, the Laguerre polynomials.

Papoulis Method

The Papoulis method of inversion involves the expansion of f(t) into a series of Legendre polynomials and uses the classical interval of orthogonality (-1,1). It is clear from this method how a particular region of the time scale may be examined by changing the scale factor γ [8].

Let us make the transformation

$$x = e^{-\gamma t} \qquad \gamma > 0 \tag{51}$$

where f(t) becomes

$$f(t) = f(-\frac{1}{\gamma} \log x) \equiv \hat{f}(x), \tag{52}$$

Making these substitutions into equation (1) we get

$$\gamma F(s) = \int_0^1 x^{(s/\gamma)-1} \hat{f}(x)dx , \tag{53}$$

By expressing s in terms of γ as $s = (2\kappa+1)\gamma$, equation (53) becomes

$$\gamma F(2\kappa\gamma+\gamma) = \int_0^1 x^{2\kappa} \hat{f}(x)dx. \tag{54}$$

Defining $\hat{f}(x)$ as an even function (i.e., $\hat{f}(x) = \hat{f}(-x)$), the function $\hat{f}(x)$ can be expanded into a series of even Legendre polynomials as

$$\hat{f}(x) = \sum_{N=o}^{\infty} C_N P_{2N}(x). \tag{55}$$

Using the fact that $P_{2N}(e^{-\gamma t})$ is an even polynomial in $e^{-\gamma t}$ of degree 2N, Papoulis shows that the transform of $P_{2N}(e^{-\gamma t})$ is of the form

$$F(s) = P_{2N}(s) = \frac{(s-\gamma)(s-3\gamma)\ldots[s-(2N-1)\gamma]}{s(s+2\gamma)\quad\ldots\quad(s+2N\gamma)} \tag{56}$$

Replacing s first by γ then 3γ, 5γ,...$(2N+1)\gamma$ ... in equation (56), gives the following system of equations to evaluate C_N:

$$\begin{aligned} \gamma F(\gamma) &= C_o \\ \gamma F(3\gamma) &= \frac{1}{3} C_o + \frac{2}{15} C_1 \\ &\vdots \\ F(2N\gamma+\gamma) &= \frac{C_o}{2N+1} + \frac{2NC_1}{(2N+1)(2N+3)} + \ldots + \frac{2N(2N-2)\ \ldots 2C_N}{(2N+1)(2N+3)\ldots(4N+1)}\ . \end{aligned} \tag{57}$$

Consequently the C_N's are determined and the function f(x) in real time is

$$f(t) = \sum_{N=o}^{\infty} C_N\ P_{2N}(e^{-\gamma t}). \tag{58}$$

Gauss-Laguerre Quadrature

The Laguerre polynomials are the unique Gauss quadrature polynomials in the interval $(0,\infty)$ with weighting function

$$w(x) = e^{-\alpha x}. \tag{59}$$

Denote the Laguerre polynomials by $L_N(x)$ where

$$L_N(x) = \pi(x)(-1)^N\ , \tag{60}$$

and the interval of expansion is $(0,\infty)$. These polynomials are defined by the Rodrigue's formula

$$L_N(\alpha x) = e^{\alpha x} \frac{d^N}{dx^N}\left(x^N\ e^{-\alpha x}\right). \tag{61}$$

It follows from equation (14) that the orthogonality condition is

$$\int_0^\infty e^{-\alpha x} L_N(\alpha x)\, L_M(\alpha x) dx = 0 \qquad M \neq N \tag{62}$$

Consequently the resursion relations are

$$L_{N+1}(x) = (1 + 2N - x)\, L_N(x) - N^2 L_{N-1}(x) , \tag{63}$$

and

$$L'_{N+1}(x) = (N+1)[L'_N(x) - L_N(x)]. \tag{64}$$

The least squares polynomial approximation to f(x) over $(0,\infty)$ with respect to the weighting function

$$w(x) = e^{-\alpha x} , \tag{65}$$

is

$$F(x) = \sum_{K=0}^{N} b_K L_K(\alpha x) , \tag{66}$$

where

$$b_K = \frac{\alpha}{(K!)^2} \int_0^\infty e^{-\alpha x} f(x)\, L_K(\alpha x) dx . \tag{67}$$

Consequently the coefficients (or weights) become

$$A_i = \int_0^\infty e^{-\alpha x} \frac{L_N(\alpha x)}{(x-x_i) L'_N(\alpha x_i)} dx . \tag{67}$$

Using the properties stated above, a similar procedure to that employed with the Legendre polynomials may be performed and an explicit inversion formula derived.

TEST CASES

A numerical technique has been developed based on shifted Legendre polynomials to invert the Laplace transform of a function. Let us now examine the accuracy obtained in utilizing this technique.

Two test functions are considered. The first function is $u(t)$ = constant = 1 where the Laplace transform is $v(s) = \frac{1}{s}$. The second test function is $u(t) = e^{-t}$ where the Laplace transform is $v(s) = \frac{1}{s+1}$. The results obtained for these test cases are tabulated

in Tables I and II. Notice that the accuracy decreases as N (number of roots) increases. However 8 - 9 figure accuracy is obtained in test cases I and 6 - 9 figure accuracy is obtained in test case II. These results indicate that the numerical inversion technique used is very accurate.

TABLE I

TEST CASE I

N	EXACT VALUE	APPROXIMATE VALUE
4	0.999999999D 00	0.100000000D 01
	0.999999999D 00	0.999999999D 00
	0.999999999D 00	0.999999999D 00
	0.999999999D 00	0.999999999D 00
7	0.999999999D 00	0.999999999D 00
	0.999999999D 00	0.100000000D 01
	0.999999999D 00	0.100000000D 01
	0.999999999D 00	0.999999999D 00
	0.999999999D 00	0.100000000D 01
	0.999999999D 00	0.999999999D 00
	0.999999999D 00	0.100000000D 01
10	0.999999999D 00	0.100000000D 01
	0.999999999D 00	0.999999999D 00
	0.999999999D 00	0.100000000D 01
	0.999999999D 00	0.100000000D 01
	0.999999999D 00	0.999999999D 00
	0.999999999D 00	0.100000000D 01
	0.999999999D 00	0.999999999D 00
	0.999999999D 00	0.100000000D 01
	0.999999999D 00	0.999999999D 00
	0.999999999D 00	0.100000000D 01
14	0.999999999D 00	0.999999966D 00
	0.999999999D 00	0.999999962D 00
	0.999999999D 00	0.100000012D 01
	0.999999999D 00	0.100000003D 01
	0.999999999D 00	0.999999988D 00
	0.999999999D 00	0.100000001D 01
	0.999999999D 00	0.999999891D 00
	0.999999999D 00	0.999999944D 00
	0.999999999D 00	0.100000004D 01
	0.999999999D 00	0.999999862D 00
	0.999999999D 00	0.100000009D 01
	0.999999999D 00	0.999999977D 00
	0.999999999D 00	0.999999962D 00
	0.999999999D 00	0.100000005D 01

TABLE II

TEST CASE II

N	EXACT VALUE	APPROXIMATE VALUE
4	0.930568155D 00	0.930568155D 00
	0.669990521D 00	0.669990521D 00
	0.330009478D 00	0.330009478D 00
	0.694318442D-01	0.694318442D-01
7	0.974553956D 00	0.974553956D 00
	0.870765592D 00	0.870765592D 00
	0.702922575D 00	0.702922575D 00
	0.499999999D 00	0.500000000D 00
	0.297077424D 00	0.297077424D 00
	0.129234407D 00	0.129234407D 00
	0.254460438D-01	0.254460438D-01
10	0.986953264D 00	0.986953264D 00
	0.932531683D 00	0.932531683D 00
	0.839704784D 00	0.839704784D 00
	0.716697697D 00	0.716697697D 00
	0.574437169D 00	0.574437169D 00
	0.425562830D 00	0.425562830D 00
	0.283302302D 00	0.283302302D 00
	0.160295215D 00	0.160295215D 00
	0.674683166D-01	0.674683163D-01
	0.130467357D-01	0.130467358D-01
14	0.993141904D 00	0.993141833D 00
	0.964217441D 00	0.964217450D 00
	0.913600657D 00	0.913600761D 00
	0.843646452D 00	0.843646317D 00
	0.757624318D 00	0.757624350D 00
	0.659556184D 00	0.659556165D 00
	0.554027474D 00	0.554027389D 00
	0.445972525D 00	0.445972587D 00
	0.340443815D 00	0.340443793D 00
	0.242375681D 00	0.242375649D 00
	0.156353547D 00	0.156353551D 00
	0.863993424D-01	0.863993391D-01
	0.357825581D-01	0.357825607D-01
	0.685809565D-02	0.685815140D-02

III. THE INVERSE PROBLEM IN TRANSIENT HEAT CONDUCTION

INTRODUCTION

The problems presented in this section are representative of typical one, two, and three dimensional geometries. They were selected because of the existence of exact solutions, necessary for comparison of results. However, more complicated geometries can be accommodated by this method of solution, but the equations and analysis become quite complex. The FORTRAN IV programs used in the solution of these problems are listed in Appendix A.

DIMENSIONLESS VARIABLES

The solution to a problem in heat conduction can always be expressed in terms of dimensionless variables. This is usually done when making numerical calculations and is adopted here because of simplification in the differential equations involved and generality in application of the solutions obtained.

The dimensionless variables used in the following problems are

$$\tau = \frac{Kt}{\ell^2} = \frac{Kt}{R^2} \quad ,$$

where $K = \frac{k}{Cp(\text{density})}$,

$$\rho = \frac{x}{\ell} = \frac{r}{R} \quad ,$$

$$\bar{u} = \frac{uK}{\beta\ell^2} = \frac{uK}{\beta R^2} \quad ,$$

$$\alpha_N = R\,\gamma_N \quad .$$

THE SEMI-INFINITE SOLID

The method of numerical inversion of the Laplace transform is now applied to the solution of the semi-infinite solid. Since there exists no physical characteristic length, dimensional variables are used in the solution to this problem.

The partial differential equation governing the temperature function is

$$\frac{\partial u}{\partial t} = K \frac{\partial^2 u}{\partial x^2} \quad , \tag{69}$$

with the boundary conditions

$$u(0,t) = \beta t \quad , \tag{70}$$

and

$$u(x,0) = 0 \quad . \tag{71}$$

The solution of this direct problem is

$$u(x,t) = 4\beta t i^2 \mathrm{erfc} \frac{x}{2\sqrt{Kt}} \quad . \tag{72}$$

Using this solution as the known temperature function at x_1 (an interior location), we can solve the inverse problem to obtain $u(0,t)$ at $x = 0$ (the surface of the solid). Here the exact solution is known as

$$u(0,t) = \beta t \quad . \tag{73}$$

Transforming the differential equation into the s plane we obtain the ordinary differential equation

$$\frac{d^2 v}{dx^2} - \frac{s}{K} v = 0 \tag{74}$$

where v is the Laplace transform of $u(x,t)$ and is a function of x and s. The solution to equation (74) is of the form

$$v(x,s) = A\, e^{-\sqrt{s/K}\, x} \tag{75}$$

Solving this equation at $x = x_1$ we find that

$$A = v(x_1,s) e^{\sqrt{s/K}\, x_1} \tag{76}$$

and therefore the solution of equation (74) becomes

$$v(x,s) = v(x_1,s)e^{-\sqrt{s/K}\,(x-x_1)} \tag{77}$$

Since the temperature function at an interior location x_1 is given in equation (72), we can solve equation (77) by taking the Laplace transform of equation (72) which then becomes

$$v(x_1,s) = \frac{\beta}{s^2}\, e^{-\sqrt{s/K}\,x_1} \tag{78}$$

Substituting equation (78) into equation (77) we have the solution in the s plane as

$$v(x_1,s) = \left(\frac{\beta}{s^2}\, e^{-\sqrt{s/K}\,x_1}\right) e^{-\sqrt{s/K}\,(x-x_1)} \tag{79}$$

At the surface of the solid x = 0, and the solution is

$$v(0,s) = \left(\frac{\beta}{s^2}\, e^{-\sqrt{s/K}\,x_1}\right) e^{\sqrt{s/K}\,x_1} \tag{80}$$

Taking the Laplace inverse of equation (80) we obtain the solution in the time plane at x = 0 (the surface of the solid).

An illustration of the semi-infinite problem is presented in Figure 1. In obtaining the results plotted in Figure 2 it was found advantageous to use a convergence factor. The convergence factor e^{-t} was added to the function u(x,t) to improve the accuracy of solution. The function u(x,t) was multiplied by e^{-t}, transformed into the s plane and subsequently inverted back into the time plane. The final answer was then multiplied by e^{t}. Note that the Laplace transform of this function is

$$L\{e^{-t}u(x,t)\} = F(s+1).$$

THE LONG CYLINDER

The basic assumption involved in the solution of the "long" cylinder is that the cylinder is sufficiently long so that end effects can be neglected. In other words conduction is considered to be only

in the radial direction (Figure 1). The governing partial differential equation for the temperature function u(r,t) is

$$\frac{\partial u}{\partial t} = K\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right), \tag{81}$$

subject to the boundary conditions

$$u(R,t) = \beta t \ , \tag{82}$$

and

$$u(r,0) = 0 \ . \tag{83}$$

In terms of dimensionless variables, the solution of the direct problem can be represented as the sum of an infinite series as

$$\bar{u}(\rho,\tau) = \tau - \frac{1-\rho^2}{4} + 2\sum_{N=1}^{\infty} e^{-\alpha_N{}^2\tau}\ \frac{J_o(\rho\alpha_N)}{\alpha_N{}^3\ J_1(\alpha_N)} \ , \tag{84}$$

where the α_N are defined by

$$J_o(\alpha_N) = 0 \ . \tag{85}$$

Using this solution as the known temperature function at ρ_1 (an interior location), we can solve the inverse problem to obtain $\bar{u}(\rho,\tau)$ at $\rho = 1$ (the surface). Here the exact solution is known as

$$\bar{u}(1,\tau) = \tau \ . \tag{86}$$

Proceeding as before we find that the solution of the inverse problem in the s plane is

$$v(1,s) = v(\rho_1,s)\ \frac{I_o\left(\sqrt{s}\right)}{I_o\left(\sqrt{s}\ \rho_1\right)} \ , \tag{87}$$

where

$$v(\rho_1,s) = \frac{1}{s^2} - \frac{1-\rho_1{}^2}{4s} + 2\sum_{N=1}^{\infty}\left(\frac{1}{\alpha_N{}^2+s}\right)\frac{J_o(\rho_1\alpha_N)}{\alpha_N{}^3\ J_1(\alpha_N)} \ . \tag{88}$$

By numerically inverting equation (87), we obtain the solution of the inverse problem at the surface ($\rho = 1$) where the exact solution is given by equation (86).

The results obtained for the "long" cylinder are plotted in Figures 5 and 6. It should be noted that as the distance from the surface is increased, the accuracy of solution decreases (Figure 5).

This is reasonable since the temperature will be dampened more as the distance is increased. Let us consider a characteristic parameter

$$\eta = \frac{1-r}{2\sqrt{Kt}} \quad .$$

If a thermal penetration depth is taken as

$$\delta = 2\sqrt{Kt} \quad ,$$

then physical significance can be given to η. Now we can examine the accuracy of solution as η is varied. In Figure 5, surface temperature is plotted versus η for δ = constant. It can be seen that as η is increased, the accuracy of solution decreases. Since the only variable in η is now r, it is obvious that as r increases, the accuracy of solution increases. By interpreting small η as "close" to the surface and large η as "far away" from the surface, we can see the effects of damping on the solution.

A convergence factor was also used in this problem to improve the accuracy. As before, e^{-t} was used as the convergence factor, and it can be seen from Figure 4 that the solution is improved. However the results obtained are not quite as good as the other problems considered. The inaccuracy introduced into this problem is due to the evaluation of an infinite series of Bessel functions.

THE SPHERE

The basic assumption involved in the solution of the sphere is axial symmetry. In other words the temperature function is not dependent on ϕ (Figure 1). The governing partial differential equation for the temperature function u(r,t) is

$$\frac{\partial u}{\partial t} = K\left(\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r}\right) , \tag{89}$$

subject to the boundary conditions

$$u(R,t) = \beta t \quad , \tag{90}$$

and

$$u(r,0) = 0 \quad . \tag{91}$$

In terms of dimensionless variables, the solution of the direct problem can be represented as the sum of an infinite series as

$$\bar{u}\ (\rho,\tau) = \tau - \frac{1-\rho^2}{6} - \frac{2}{\pi^3\rho} \sum_{N=1}^{\infty} \frac{(-1)^N}{N^3}\ e^{-N^2\pi^2\tau} \sin(N\pi\rho) . \tag{92}$$

Using this solution as the known temperature function at ρ_1 (an interior location), we can solve the inverse problem to obtain $\bar{u}(\rho,\tau)$ at $\rho = 1$ (the surface). Here the exact solution is known as

$$\bar{u}(1,\tau) = \tau \ . \tag{93}$$

Proceeding as before we find that the solution of the inverse problem in the s plane is

$$v(1,s) = v(\rho_1,s)\ \frac{\rho_1 \sinh\ (\sqrt{s})}{\sinh\ (\sqrt{s}\ \rho_1)} \quad , \tag{94}$$

where

$$v(\rho_1,s) = \frac{1}{s^2} - \frac{1-\rho_1{}^2}{6s} - \frac{2}{\pi^3\rho_1} \sum_{N=1}^{\infty} \frac{(-1)^N}{N^3} \left(\frac{1}{N^2\pi^2+s}\right) \sin(N\pi\rho_1) . \tag{95}$$

By numerically inverting equation (94), we obtain the solution of the inverse problem at the surface ($\rho = 1$) where the exact solution is given by equation (93).

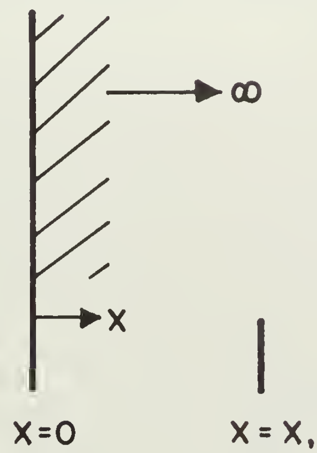
The results obtained for the sphere are plotted in Figure 7. Here as before a convergence factor e^{-t} was used to improve the solution. The results shown in Figure 7 are in very good agreement with the known exact solution.

INVERSE PROBLEMS IN TRANSIENT HEAT CONDUCTION

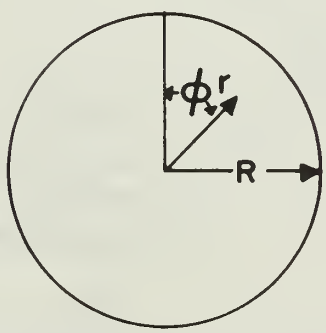
SEMI-INFINITE SOLID

ASSUMPTION

CONDUCTION OCCURS ONLY IN X-DIRECTION



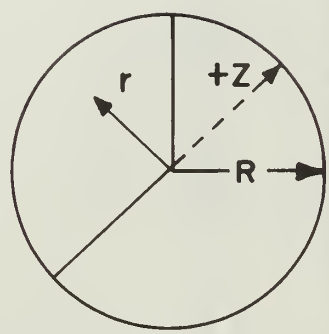
SPHERE



ASSUMPTION

TEMPERATURE FUNCTION DOES NOT DEPEND ON ϕ. THERE– FORE CONDUCTION OCCURS IN THE RADIAL DIRECTION.

"LONG" CYLINDER



ASSUMPTION

END EFFECTS ARE NEGLECTED AND THUS CONDUCTION OCCURS ONLY IN THE RADIAL DIRECTION.

FIGURE 1

SURFACE TEMPERATURE vs. TIME
FOR
SEMI-INFINITE SOLID.

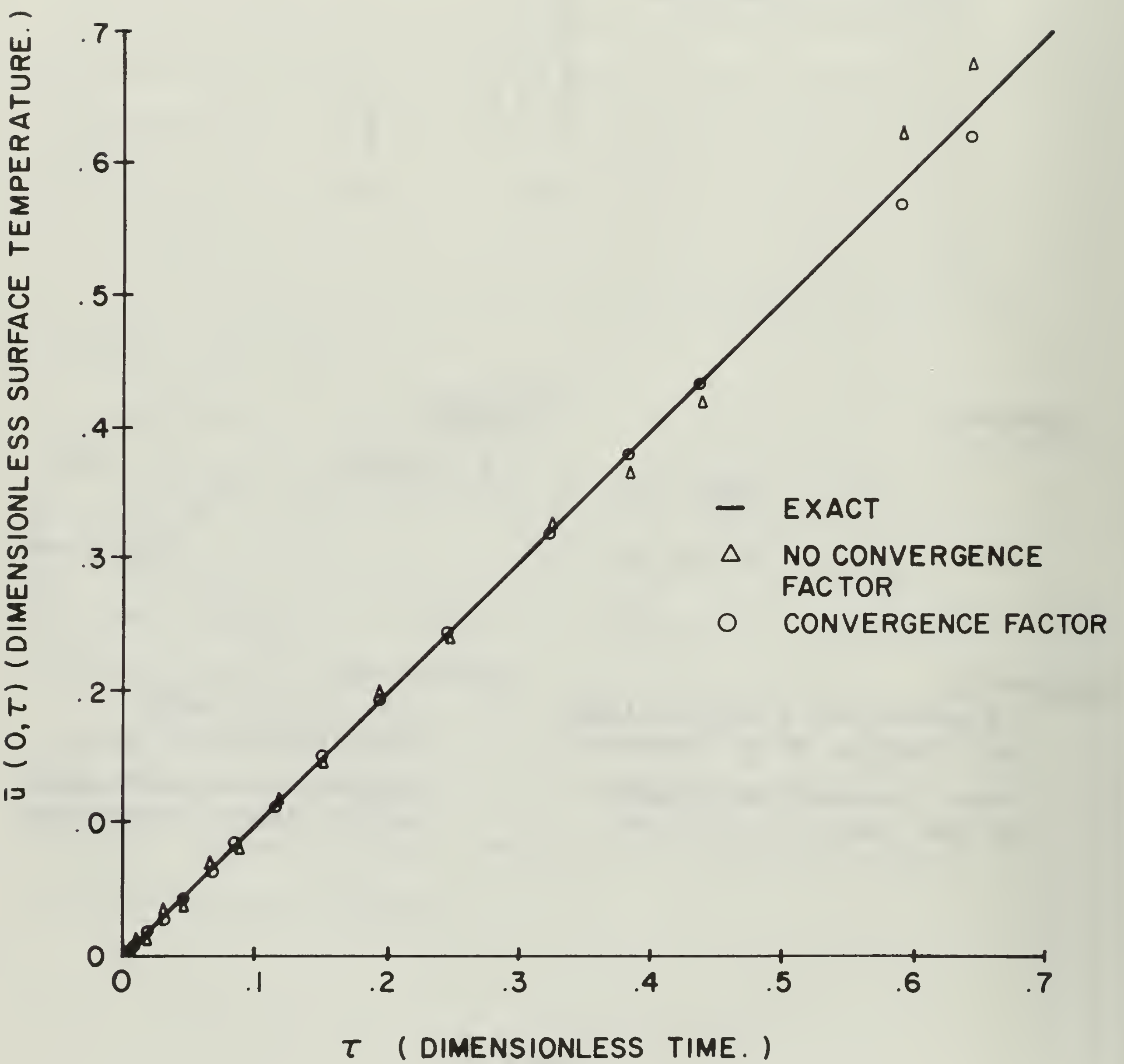


FIGURE 2.

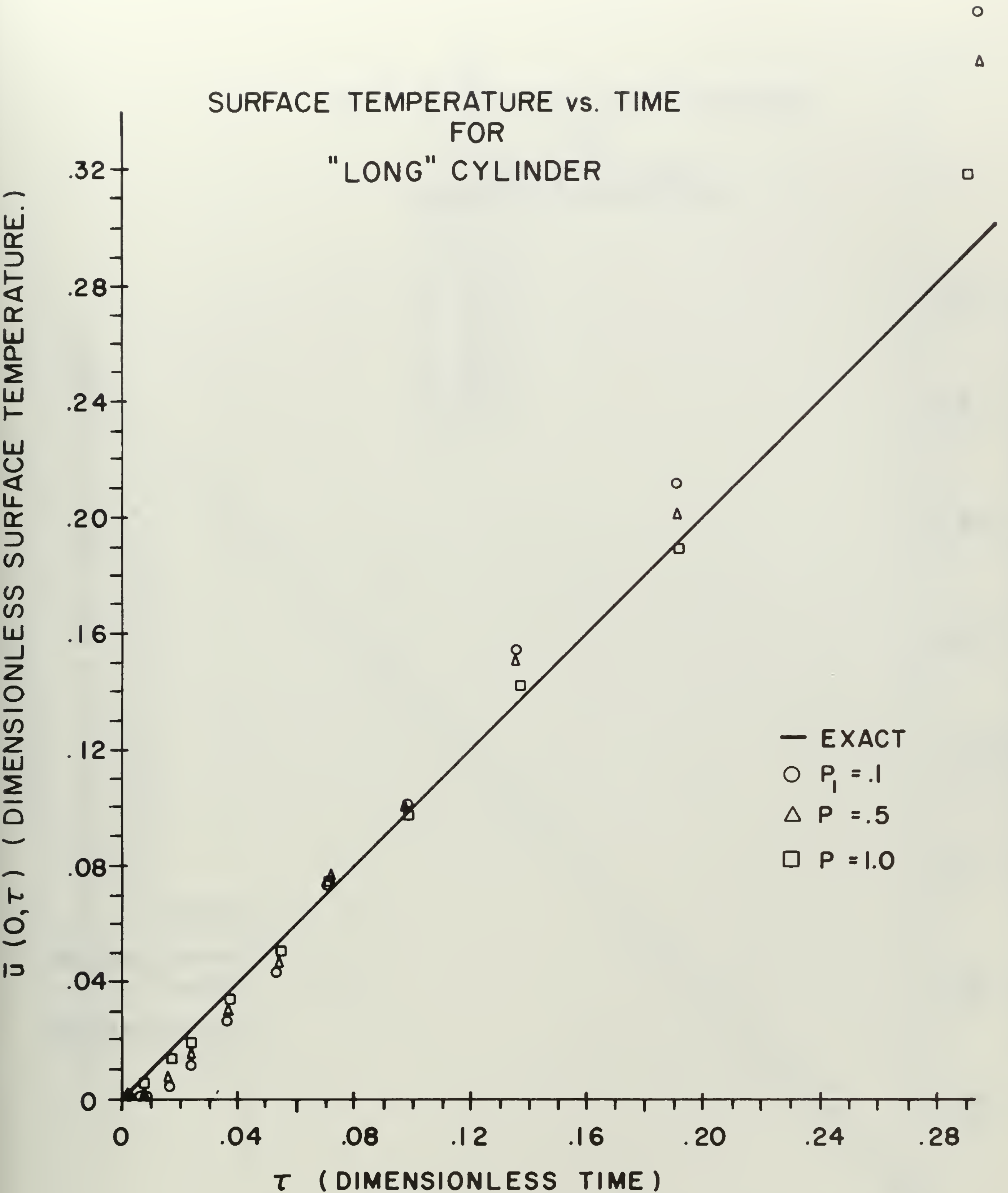

SURFACE TEMPERATURE vs. TIME
FOR
"LONG" CYLINDER
$\bar{u}(0,\tau)$ (DIMENSIONLESS SURFACE TEMPERATURE.)
.32
.28
.24
.20
.16
.12
.08
.04
0
.04
.08
.12
.16
.20
.24
.28
τ (DIMENSIONLESS TIME)
EXACT
P_1 = .1
P = .5
P = 1.0

FIGURE 3.

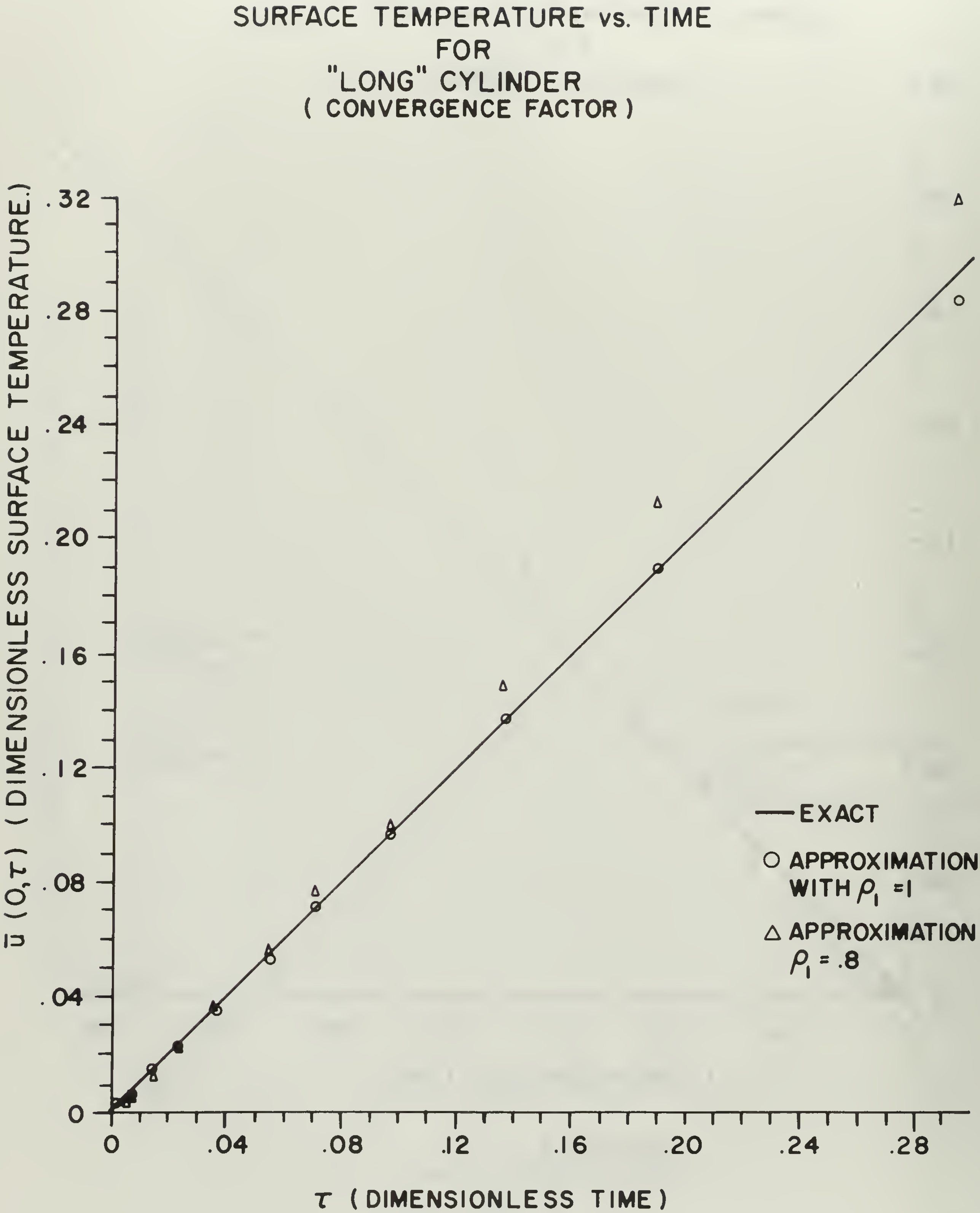
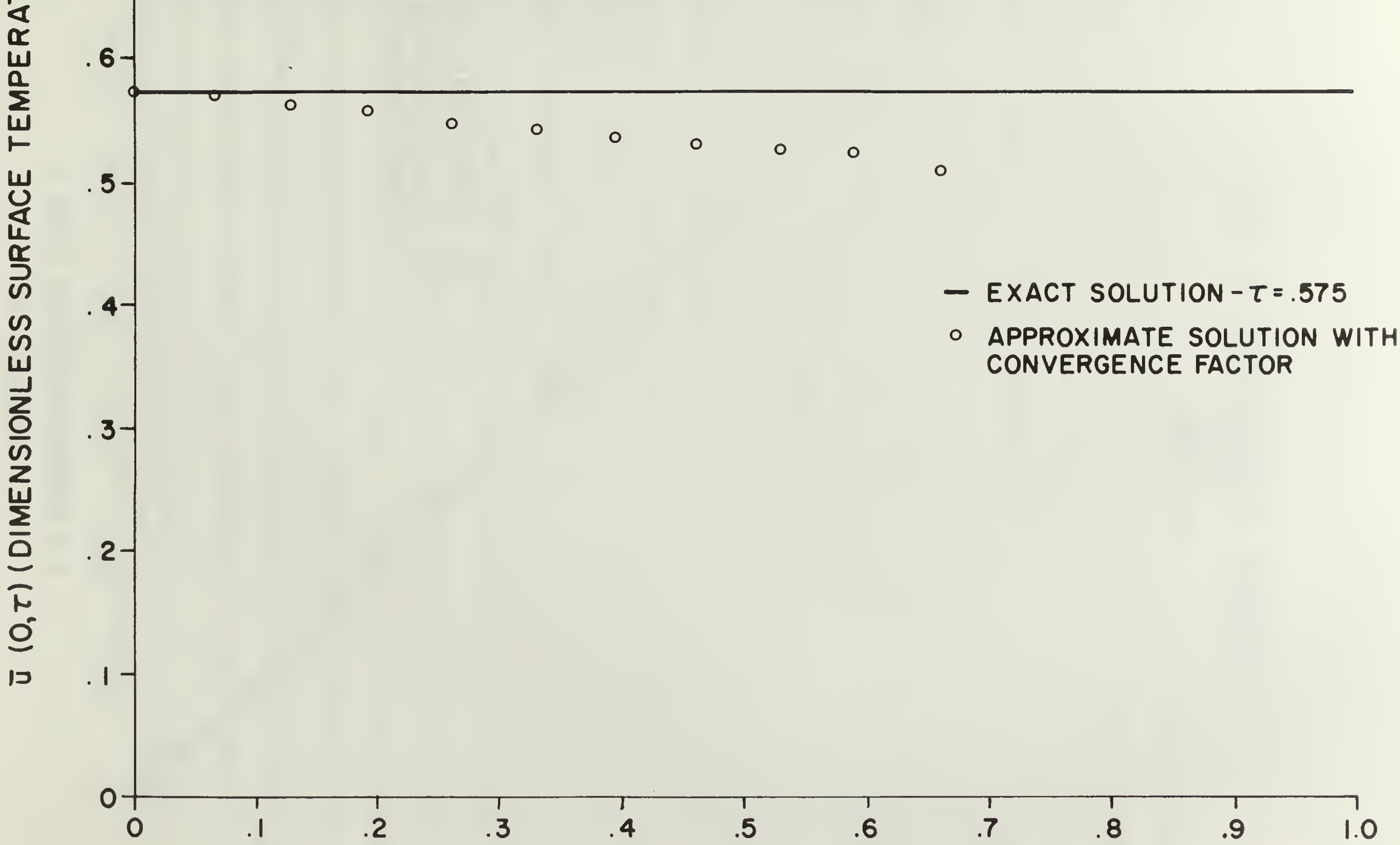

SURFACE TEMPERATURE vs. TIME
FOR
"LONG" CYLINDER
(CONVERGENCE FACTOR)
$\bar{u}$ (0,τ) (DIMENSIONLESS SURFACE TEMPERATURE.)
.32
.28
.24
.20
.16
.12
.08
.04
0
0
.04
.08
.12
.16
.20
.24
.28
τ (DIMENSIONLESS TIME)
EXACT
O APPROXIMATION WITH ρ_1 =1
△ APPROXIMATION ρ_1 = .8

FIGURE 4.

SURFACE TEMPERATURE vs. n FOR "LONG" CYLINDER

(CONVERGENCE FACTOR AND K= 1)



$n = \frac{\rho_1}{2\sqrt{\tau}}$

FIGURE 5

SURFACE TEMPERATURE vs. TIME
FOR
SPHERE

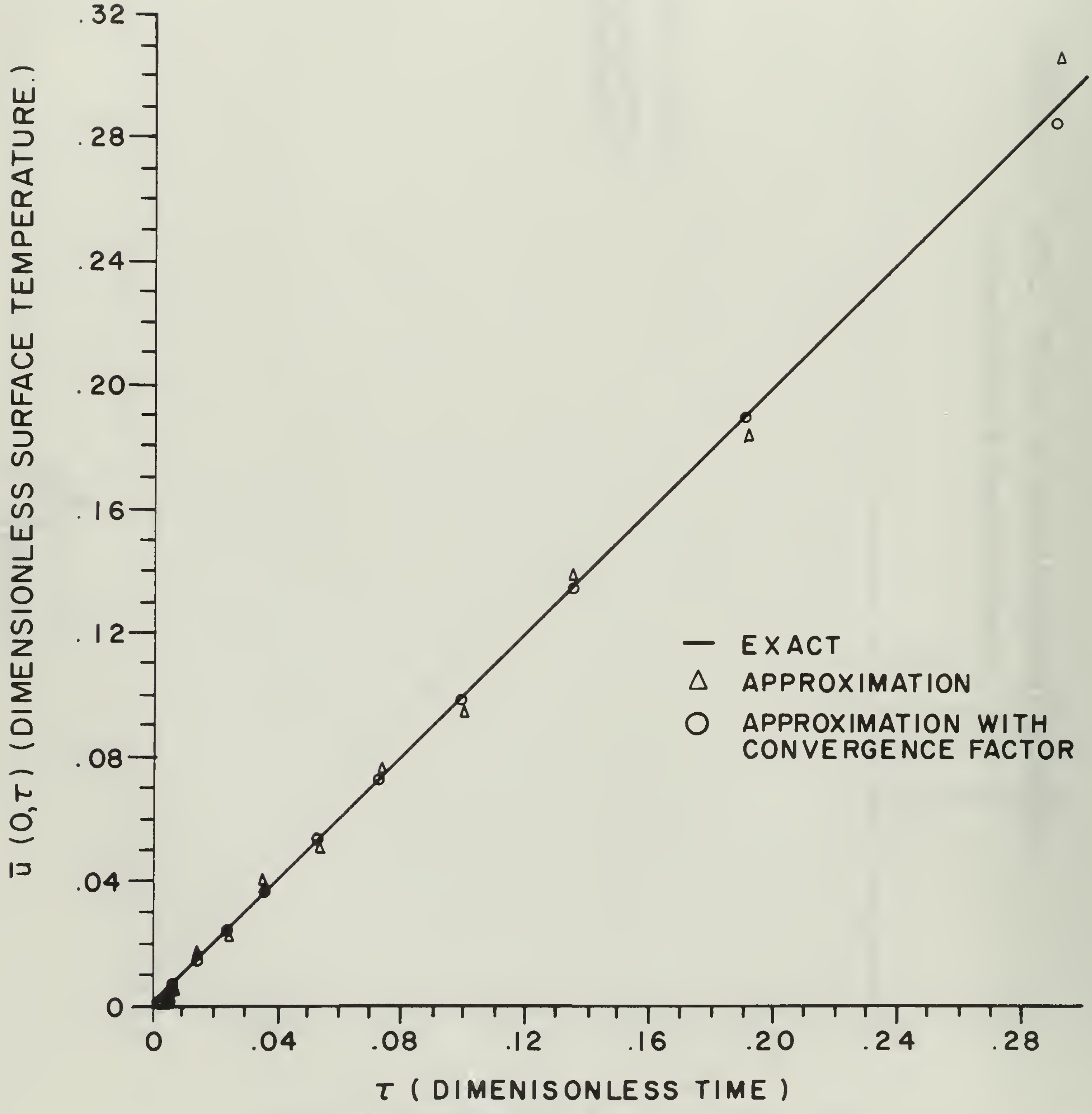


FIGURE 6.

DISCUSSION OF RESULTS

In the determination of surface temperature by the method of numerical inversion of the Laplace transform, it was found necessary to utilize a convergence factor to improve accuracy. The function u(x,t) was multiplied by e^{-t}, transformed into the s plane, and numerically inverted into the time plane. Then the function obtained was multiplied by e^{t} to obtain the desired surface temperature function u(0,t). This method caused the approximation to converge to the known exact solution. In the semi-infinite solid and the sphere, the results obtained from the use of the convergence factor were very good. In the "long" cylinder the results obtained were not quite as good. The inaccuracy introduced in this problem is due to the evaluation of an infinite series of Bessel functions.

The damping effects were considered for the case of the "long" cylinder. It was found that as the distance from the surface was increased, the accuracy of the solution decreased. This was a reasonable result, since it was logical to expect the solution to lag as the distance from the surface was increased.

The use of time scaling (mentioned in section II) was only incorporated in the semi-infinite solid FORTRAN IV program, but it can be included in the other problems. Although the initial temperature distribution was taken as zero, any reasonable initial temperature distribution can be handled with this method. Such a change would simply result in more terms to be inverted. Also it should be noted that although the solution was obtained for the surface temperature, heat flux at the surface can also be determined.

CONCLUSIONS

Three geometries were considered in the solution to the inverse problem in transient heat conduction. These geometries, the semi-infinite solid, "long" cylinder, and sphere, are typical of one, two, and three dimensional problems respectively (it should be noted here that an analogous approach can be made in the solution to the finite plane bounded by two parallel planes). Solutions were obtained which agreed very well to the known exact solution.

Several parameters were investigated with respect to the effects on the accuracy of solution. It was found in all cases that utilizing a convergence factor caused the solution to better approximate the known exact solution. The damping effect was considered in the "long" cylinder problem. As expected it was found that as ρ_1 increased (distance from the surface decreased) the accuracy of solution increased.

Therefore the solutions obtained indicate that the method of numerically inverting the Laplace transform provides an accurate approximation to the known exact solutions.

IV. SUMMARY

The numerical technique involved in numerically inverting the Laplace transform was methodically developed. Elementary theorems of the Laplace transform were introduced, and the instability of the inverse of the Laplace transform was briefly discussed. Gaussian quadrature methods were subsequently introduced and developed. The shifted Legendre polynomials were then examined for several reasons. It was found that the coefficients of the shifted polynomials were integers, and the region of orthogonality (0,1) was found to be most convenient in the following development. Then, an explicit inverse matrix was found which would be used later to solve the inverse problem in transient heat conduction. Two useful numerical properties were discussed. Interlacing of the roots (and therefore times) was found useful in interpolation of the results, while change of the time scale was found useful in extending the range of times. Two alternate methods of solution (Papoulis and Gauss-Laguerre quadrature) that could be used to obtain an explicit inverse matrix were subsequently introduced. Finally, the explicit inverse matrix was tested for accuracy. The two test cases employed were for the functions $u(t) = 1$, and $u(t) = e^{-t}$. The accuracy was found to be very good for both cases.

Three geometries were considered in the solution of the inverse problem in transient heat conduction. These problems, the semi-infinite solid, "long" cylinder and sphere, are typical of one, two, and three dimensional problems. They were chosen because of the existence of exact solutions. It should be noted that the results obtained were for discrete points and not for a continuous spectra of

information. Consequently this method is applicable to problems where discrete information is known (i.e., data from experiment).

Several parameters were examined with respect to influence on the accuracy of solution. A convergence factor e^{-t} was found to improve the accuracy of solution in all cases considered. It was found that as the distance from the surface was increased, the accuracy of solution decreased. This was considered reasonable since damping effects should be more noticeable at greater distances from the surface.

Consequently the solution to the inverse problem in transient heat conduction by numerical inversion of the Laplace transform was found to yield accurate results for semi-infinite solid, "long" cylinder, and sphere.

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APPENDIX A

FORTRAN IV PROGRAMS

1. The Semi-infinite Solid

The following program is listed:

SEMI-INFINITE PLANE SOLUTION USING DIMENSIONLESS VARIABLES IN ONE-DIMENSIONAL SYSTEM.

The following subroutines are needed:

INPUT	Subroutine
EXPINV	Subroutine

FORTRAN IV PROGRAM SYMBOLS

MAIN PROGRAMS

A	Explicit inverse matrix
R	Roots of the shifted Legendre polynomials (x_i)
W	Weights of the shifted Legendre polynomials (A_i^*)
T1	Values of time ($t_i = -\log x_i$)
TAU	Non-dimensional time ($\tau = \frac{Kt}{\ell^2} = \frac{Kt}{R^2}$)
RO	Distance (ρ)
RO1	Interior location (ρ_1)
APPR	Approximate value of v(x,s)
XK	Reciprocal of K
EXACT	Known exact solution
SCALE	Scale factor (a)
C	Value of $e^{-t}\ \bar{u}(\rho,\tau)$
ALFA	Roots of the equation $J_o(\alpha_N) = 0$
ARG	Argument of the function $J_o(\alpha_N\rho_1)$
ARQ	Argument of the function $I_o(\sqrt{s}\ \rho_1)$
ARP	Argument of the function $I_o(\sqrt{s}\ )$
BESSO	Resultant value of $J_o(\alpha_N\rho_1)$
BESS1	Resultant value of $J_1(\alpha_N)$
BISS	Resultant value of $I_o(\sqrt{s}\ )$
BISS1	Resultant value of $I_o(\sqrt{s}\ \rho_1)$
ADA	Value of η
N	Number of roots

EXPINV SUBROUTINE

C	Legendre coefficients
B	Coefficients of the polynomials $P_N^*(x)/(x-x_i)$
P	Legendre derivatives

```
C
C     SOLUTION OF THE INVERSE PROBLEM IN TRANSIENT HEAT CONDUCTION
C
C     BOUNDARY CONDITIONS ARE:
C                           1. INITIAL TEMPERATURE DISTRIBUTION IS ZERO.
C                           2. SURFACE TEMPERATURE DISTRIBUTION IS:  U(1,TAU)=TAU
C
C      I. SEMI-INFINITE PLANE SOLUTION USING DIMENSIONLESS VARIABLES
C      IN ONE-DIMENSIONAL  SYSTEM.
C
           DIMENSION T1(15),A(15,15),R(15),W(15),TAU(15),C(15),RO(11),APP
     *R(15),EXACT(15),F(20)
           REAL*8 A,R,W,C,T1,TAU,APPR,EXACT,F,XK,RO,ADA(20),S,SCALE,Q,EPP
     *R(20),RO1(20)
           DO 100 JJ=2,15
           CALL INPUT(N,R,W,T1)
           CALL EXPINV(N,R,W,A)
           WRITE(6,106)
           DO 20 I=1,N
   20      WRITE(6,102)R(I),W(I)
           SCALE=2.0D+00
           XK=1.0D+02
           PI=3.14159265
           DO 4 II=1,N
    4      TAU(II)=SCALE*T1(II)/XK
           IP=1
           RO1(IP)=0.0D+00
           IR=1
           RO(IR)=0.0D+00
   24      CONTINUE
           DO 40 IS=1,N
           Q=IS+1
           S=Q/SCALE
           EPPR(IS)=DEXP(-DSQRT(S*XK)*RO1(IP))/S**2
           APPR(IS)=EPPR(IS)*DEXP(-DSQRT(S*XK)*(RO(IR)-RO1(IP)))/(XK*SCAL
     *E)
           EXACT(IS)=TAU(IS)
           ADA(IS)=RO1(IP)/(2.0D+00*DSQRT(T1(IS)/XK))
   40      CONTINUE
           DO 61 I=1,N
           C(I)=0.0D+00
           DO 61 J=1,N
   61      C(I)=C(I)+A(I,J)*APPR(J)
           DO 62 J=1,N
   62      F(J)=C(J)*DEXP(T1(J))
           WRITE(6,83)N
           WRITE(6,80)RO1(IP)
           WRITE(6,60)
           WRITE(6,70)(T1(I),F(I),EXACT(I),ADA(I),I=1,N)
           IP=IP+1
           RO1(IP)=RO1(IP-1)+0.1D+00
           IF(RO1(IP).LT.1.000) GO TO 24
  100      CONTINUE
           CALL EXIT
   60      FORMAT(//,11X,'TIME',14X,'APPROXIMATE VALUE',15X,'EXACT VALUE'
     *,20X,'ADA',//)
   70      FORMAT(4(2X,D24.17),/)
   80      FORMAT(//,3X,'THE VALUE OF RO1 IS  ',F6.4,//)
   83      FORMAT(//,3X,'THE VALUE OF N IS  ',I5,//)
  102      FORMAT(//,2(2X,D24.17),/)
  106      FORMAT(//,10X,'ROOTS',13X,'WEIGHTS',//)
  108      FORMAT(//,5X,'T =-LOG R',20X,/,D24.17,/)
           END
```

2. The "Long" Cylinder

The following program is listed:

TWO-DIMENSIONAL "LONG" CYLINDER SOLUTION.

The following subroutines are needed:

INPUT	subroutine
EXPINV	subroutine
BESI	subroutine
BESJ	subroutine

```
SOLUTION OF THE INVERSE PROBLEM IN TRANSIENT HEAT CONDUCTION

BOUNDARY CONDITIONS ARE:
                1. INITIAL TEMP DISTRIBUTION IS ZERO.
                2. SURFACE TEMP DISTRIBUTION IS : U(1,TAU)=TAU.
 III. TWO-DIMENSIONAL "LONG"CYLINDER SOLUTION.

      DIMENSION T1(15),A(15,15),R(15),W(15),TAU(15),C(15),RO(15),APP
     *R(15),EXACT(15),ARG(20),SUM(15),ALFA(15),ARP(20),ARQ(20),RO1(20)
      REAL*8 A,R,W,C,T1,TAU,APPR,EXACT,ALFA,SUM,BESS0,BESS1,BISS,BIS
     *S1,ARG,ARQ,ARP,S,XK,F(20)
       DO 2 I=1,10
 2    READ(5,110)ALFA(I)
      DO 100 JJ=2,15
      CALL INPUT(N,R,W,T1)
      CALL EXPINV(N,R,W,A)
      WRITE(6,106)
      DO 20 I=1,N
 20   WRITE(6,102)R(I),W(I)
      XK=1.6D+01
      DO 4 II=1,N
 4    TAU(II)=T1(II)/XK
      PI=3.14159265
      IR=1
      RO(IR)=1.0D+00
      IP=1
      RO1(IP)=0.0D+00
 24   CONTINUE
      DO 90 IS=1,N
      SUM(IS)=0.0D+00
      S=IS+1
      DO 6 K=1,10
      ARG(K)=RO1(IP)*ALFA(K)
      CALL BESJ(ARG,0,BESS0,1.0D-12,IER)
      CALL BESJ(ALFA,1,BESS1,1.0D-12,IER)
 6    SUM(IS)=SUM(IS)+(1.0D+00/(ALFA(K)**2+S))*BESS0/(ALFA(K)**3*BES
     *S1)
      ARP(IS)=DSQRT(S)
      ARQ(IS)=RO1(IP)*DSQRT(S)
      CALL BESI(ARQ,0,BISS1,IER)
      CALL BESI(ARP,0,BISS,IER)
      APPR(IS)=((1.0D+00/S**2)-((1.0D+00-RO1(IP)**2)/(4.0D+00*S))+2.
     *0D+00*SUM(IS))*BISS/(BISS1*XK)
      EXACT(IS)=TAU(IS)
 90   CONTINUE
      DO 50 I=1,N
      C(I)=0.0D+00
      DO 50 J=1,N
 50   C(I)=C(I)+A(I,J)*APPR(J)
      DO 62 J=1,N
 62   F(J)=C(J)*DEXP(T1(J))
      WRITE(6,83)N
      WRITE(6,80)RO1(IP)
      WRITE(6,60)
      WRITE(6,70)(TAU(I),F(I),EXACT(I),I=1,N)
      IP=IP+1
      RO1(IP)=RO1(IP-1)+0.1D+00
      IF(RO1(IP).LT.1.000) GO TO 24
 100  CONTINUE
      CALL EXIT
 60   FORMAT(//,11X,'TIME',14X,'APPROXIMATE VALUE',15X,'EXACT VALUE'
     *,//)
 70   FORMAT(3(2X,D24.17),/)
 80   FORMAT(//,3X,'THE VALUE OF RO1 IS  ',F6.4,//)
 83   FORMAT(//,3X,'THE VALUE OF N IS  ',I5,//)
 102  FORMAT(//,2(2X,D24.17),/)
 106  FORMAT(//,10X,'ROOTS',13X,'WEIGHTS',//)
 108  FORMAT(//,5X,'T =-LOG R',20X,/,D24.17,/)
 110  FORMAT(D13.6)
      END
```

3. The Sphere

The following program is listed:

THREE-DIMENSIONAL SPHERE SOLUTION.

The following subroutines are needed:

INPUT	subroutine
EXPINV	subroutine

```
SOLUTION OF THE INVERSE PROBLEM IN TRANSIENT HEAT CONDUCTION

BOUNDARY CONDITIONS ARE:
                1. INITIAL TEMPERATURE DISTRIBUTION IS ZERO.
                2. SURFACE TEMPERATURE DISTRIBUTION IS:  U(1,TAU)=TAU

 IV. THREE-DIMENSIONAL SPHERE SOLUTION

      DIMENSION T1(15),A(15,15),R(15),W(15),TAU(15),C(15),RO(11),APP
     *R(15),EXACT(15),SUM(15),RO1(21)
      REAL*8 A,R,W,C,T1,TAU,APPR,EXACT,SUM,F(20)
      DO 100 JJ=2,15
      CALL INPUT(N,R,W,T1)
      CALL EXPINV(N,R,W,A)
      WRITE(6,106)
      DO 20 I=1,N
   20 WRITE(6,102)R(I),W(I)
      XK=16.000000000000000
      DO 4 II=1,N
    4 TAU(II)=T1(II)/XK
      PI=3.14159265
      IR=1
      RO(IR)=1.0D+00
      IP=1
      RO1(IP)=0.0D+00
   24 CONTINUE
      DO 90 IS=1,N
      SUM(IS)=0.0D+00
      DO 6 K=1,7
      S=IS+1
    6 SUM(IS)=SUM(IS)+((-1)**K/K**3)*(1.0D+00/(K**2*PI**2+S))*SIN(K*
     *PI*RO1(IP))
      APPR(IS)=((1.0D+00/S**2)-((1.0D+00-RO1(IP)**2)/(6.0D+00*S))-(2
     *.0D+00/(PI**3*RO1(IP)))*SUM(IS))*RO1(IP)*SINH(SQRT(S))/(SINH(SQRT(
     *S)*RO1(IP))*XK)
      EXACT(IS)=TAU(IS)
   90 CONTINUE
      DO 50 I=1,N
      C(I)=0.0D+00
      DO 50 J=1,N
   50 C(I)=C(I)+A(I,J)*APPR(J)
      DO 62 J=1,N
   62 F(J)=C(J)*DEXP(T1(J))
      WRITE(6,80)RO1(IP)
      WRITE(6,60)
      WRITE(6,70)(TAU(I),F(I),EXACT(I),I=1,N)
      IP=IP+1
      RO1(IP)=RO1(IP-1)+0.1D+00
      IF(RO1(IP).LT.1.000) GO TO 24
  100 CONTINUE
      CALL EXIT
   60 FORMAT(//,11X,'TIME',14X,'APPROXIMATE VALUE',15X,'EXACT VALUE'
     *,//)
   70 FORMAT(3(2X,D24.17),/)
   80 FORMAT(//,3X,'THE VALUE OF RO1 IS ',F6.4,//)
  102 FORMAT(//,2(2X,D24.17),/)
  106 FORMAT(//,10X,'ROOTS',13X,'WEIGHTS',//)
  108 FORMAT(//,5X,'T =-LOG R',20X,/,D24.17,/)
      END
```

4. Subroutines

The following subroutines are listed:

INPUT	subroutine
EXPINV	subroutine
BESI	subroutine*
BESJ	subroutine*

*"Subroutine BESJ" and "subroutine BESI," System/360 Scientific Subroutine Package (360A-CM-03X) Version II Programmer's Manual. White Plains, New York: International Business Machines Corporation, 1967, pp. 156, 157, 159.

```
C
C
      SUBROUTINE EXPINV(N,R,W,A)
      DIMENSION R(15),W(15),A(15,15),C(17,17),B(15,15),P(15)
      REAL*8 R,W,A,C,X1,X2,X,XI,B,PZERO,PREV2,PREV1,Z1,Z2,Z3,PREV3,P
     *,DENOM
C
C   LEGENDRE COEFFICIENTS
C
      NPLUS=N+1
      DO 10 M=1,NPLUS
 10   C(M,1)=1.0D+00
      DO 20 K=1,NPLUS
      DO 20 M=1,NPLUS
      IF(K.GT.M)  C(M,K)=0.0D+00
 20   CONTINUE
      DO 40 K=1,NPLUS
 15   DO 40 M=1,NPLUS
      IF(K.GT.M)  GO TO 40
      C(M+1,K+1)=((2*M-1)*C(M,K+1)-(M-1)*C(M-1,K+1)-(4*M-2)*C(M,K))/
     *M
 40   CONTINUE
C
C   B COEFFICIENTS
C
      DO 80 I=1,N
      B(I,N)=C(N+1,N+1)
 80   CONTINUE
      DO 90 I=1,N
      XI=R(I)
      K=N
 50   K=K-1
      B(I,K)=C(N+1,K+1)+XI*B(I,K+1)
      IF(K.NE.1) GO TO 50
 90   CONTINUE
C
C   LEGENDRE DERIVATIVES
C
      NMINUS=N-1
      DO 180 I=1,N
      X=R(I)
      PZERO=1.0D 00-2.0D 00*X
      PREV2=PZERO
      PREV1=1.0D 00
      DO 150 M=2,NMINUS
      Z1=M-1
      Z2=2*M-1
      Z3=M
      PREV3=-(Z1*PREV1-Z2*PZERO*PREV2)/Z3
      PREV1=PREV2
 150  PREV2=PREV3
      Z1=N
      Z2=2.0D 00*X*(X-1.0D 00)
 180  P(I)=PREV3*Z1/Z2
C
C   EXPLICIT INVERSE MATRIX
C
      DO 220 M=1,N
      DENOM=W(M)*P(M)
      DO 220 K=1,N
 220  A(M,K)=B(M,K)/DENOM
      RETURN
      END
```

```
      SUBROUTINE INPUT(N,R,W,T)
      DIMENSION R(15),W(15),T(15)
      REAL*8 R,W,T
      READ(5,100)N
      DO 10 I=1,N
10    READ(5,101)R(I)
      DO 11 I=1,N
11    READ(5,101)W(I)
      DO 12 I=1,N
12    T(I)=-DLOG(R(I))
100   FORMAT(I5)
101   FORMAT(D24.17)
      RETURN
      END
```

```
C     .......................................................................
C
C        SUBROUTINE BESI
C
C        PURPOSE
C           COMPUTE THE I BESSEL FUNCTION FOR A GIVEN ARGUMENT AND ORDER
C
C        USAGE
C           CALL BESI(X,N,BI,IER)
C
C        DESCRIPTION OF PARAMETERS
C           X   -THE ARGUMENT OF THE I BESSEL FUNCTION DESIRED
C           N   -THE ORDER OF THE I BESSEL FUNCTION DESIRED
C           BI  -THE RESULTANT I BESSEL FUNCTION
C           IER-RESULTANT ERROR CODE WHERE
C               IER=0 NO ERROR
C               IER=1 N IS NEGATIVE
C               IER=2 X IS NEGATIVE
C               IER=3 UNDERFLOW, BI .LT. 1.E-69, BI SET TO 0.0
C               IER=4 OVERFLOW, X .GT. 170 WHERE X .GT. N
C
C        REMARKS
C           N AND X MUST BE .GE. ZERO
C
C        SUBROUTINES AND FUNCTION SUBPROGRAMS REQUIRED
C           NONE
C
C        METHOD
C           COMPUTES I BESSEL FUNCTION USING SERIES OR ASYMPTOTIC
C           APPROXIMATION DEPENDING ON RANGE OF ARGUMENTS.
C
C     .......................................................................
C
      SUBROUTINE BESI(X,N, BI,IER)
          REAL*8 X,XX,TERM,BI,TOL
C
C     CHECK FOR ERRORS IN N AND X AND EXIT IF ANY ARE PRESENT
C
      IER=0
      BI=1.0
      IF(N)150,15,10
   10 IF(X)160,20,20
   15 IF(X)160,17,20
   17 RETURN
C
C     DEFINE TOLERANCE
C
   20     TOL=1.0D-12
C
C     IF ARGUMENT GT 12 AND GT N, USE ASYMPTOTIC FORM
C
      IF(X-12.)40,40,30
   30 IF(X-FLOAT(N))40,40,110
C
C     COMPUTE FIRST TERM OF SERIES AND SET INITIAL VALUE OF THE SUM
C
   40 XX=X/2.
   50 TERM=1.0
      IF(N) 70,70,55
   55 DO 60 I=1,N
      FI=I
          IF(DABS(TERM)-1.D-68)56,60,60
   56 IER=3
      BI=0.0
      RETURN
   60 TERM=TERM*XX/FI
   70 BI=TERM
      XX=XX*XX
C
C     COMPUTE TERMS, STOPPING WHEN ABS(TERM) LE ABS(SUM OF TERMS)
C     TIMES TOLERANCE
C
      DO 90 K=1,1000
```

```
          IF(DABS(TERM)-DABS(BI*TOL))100,100,80
   80 FK=K*(N+K)
      TERM=TERM*(XX/FK)
   90 BI=BI+TERM
C
C     RETURN BI AS ANSWER
C
  100 RETURN
C
C     X GT 12 AND X GT N, SO USE ASYMPTOTIC APPROXIMATION
C
  110 FN=4*N*N
      IF(X-170.0)115,111,111
  111 IER=4
      RETURN
  115 XX=1./(8.*X)
      TERM=1.
      BI=1.
      DO 130 K=1,30
          IF(DABS(TERM)-DABS(TOL*BI))140,140,120
  120 FK=(2*K-1)**2
      TERM=TERM*XX*(FK-FN)/FLOAT(K)
  130 BI=BI+TERM
C
C     SIGNIFICANCE LOST AFTER 30 TERMS, TRY SERIES
C
      GO TO 40
  140 PI=3.141592653
          BI=BI*DEXP(X)/DSQRT(2.*PI*X)
      GO TO 100
  150 IER=1
      GO TO 100
  160 IER=2
      GO TO 100
      END
```

```
C
C     ..................................................................
C
C        SUBROUTINE BESJ
C
C        PURPOSE
C           COMPUTE THE J BESSEL FUNCTION FOR A GIVEN ARGUMENT AND ORDE
C
C        USAGE
C           CALL BESJ(X,N,BJ,D,IER)
C
C        DESCRIPTION OF PARAMETERS
C           X  -THE ARGUMENT OF THE J BESSEL FUNCTION DESIRED
C           N  -THE ORDER OF THE J BESSEL FUNCTION DESIRED
C           BJ -THE RESULTANT J BESSEL FUNCTION
C           D  -REQUIRED ACCURACY
C           IER-RESULTANT ERROR CODE WHERE
C              IER=0  NO ERROR
C              IER=1  N IS NEGATIVE
C              IER=2  X IS NEGATIVE OR ZERO
C              IER=3  REQUIRED ACCURACY NOT OBTAINED
C              IER=4  RANGE OF N COMPARED TO X NOT CORRECT (SEE REMARKS
C
C        REMARKS
C           N MUST BE GREATER THAN OR EQUAL TO ZERO, BUT IT MUST BE
C           LESS THAN
C              20+10*X-X** 2/3   FOR X LESS THAN OR EQUAL TO 15
C              90+X/2            FOR X GREATER THAN 15
C
C        SUBROUTINES AND FUNCTION SUBPROGRAMS REQUIRED
C           NONE
C
C        METHOD
C           RECURRENCE RELATION TECHNIQUE DESCRIBED BY H. GOLDSTEIN AND
C           R.M. THALER,@RECURRENCE TECHNIQUES FOR THE CALCULATION OF
C           BESSEL FUNCTIONS@,M.T.A.C.,V.13,PP.102-108 AND I.A. STEGUN
C           AND M. ABRAMOWITZ,@GENERATION OF BESSEL FUNCTIONS ON HIGH
C           SPEED COMPUTERS@,M.T.A.C.,V.11,1957,PP.255-257
C
C     ..................................................................
C
      SUBROUTINE BESJ(X,N,BJ,D,IER)
        REAL*8 FM1,FM,BMK,BJ,ALFA,BPREV,D
C
      BJ=.0
      IF(N)10,20,20
   10 IER=1
      RETURN
   20 IF(X)30,30,31
   30 IER=2
      RETURN
   31 IF(X-15.)32,32,34
   32 NTEST=20.+10.*X-X** 2/3
      GO TO 36
   34 NTEST=90.+X/2.
   36 IF(N-NTEST)40,38,38
   38 IER=4
      RETURN
   40 IER=0
      N1=N+1
      BPREV=.0
C
C     COMPUTE STARTING VALUE OF M
C
      IF(X-5.)50,60,60
   50 MA=X+6.
      GO TO 70
   60 MA=1.4*X+60./X
   70 MB=N+IFIX(X)/4+2
      MZERO=MAX0(MA,MB)
C
C     SET UPPER LIMIT OF M
C
      MMAX=NTEST
```

```
130 DO 190 M=MZERO,MMAX,3
C
C     SET F(M),F(M-1)
C
      FM1=1.0E-28
      FM=.0
      ALPHA=.0
      IF(M-(M/2)*2)120,110,120
  110 JT=-1
      GO TO 130
  120 JT=1
  130 M2=M-2
      DO 160 K=1,M2
      MK=M-K
      BMK=2.*FLOAT(MK)*FM1/X-FM
      FM=FM1
      FM1=BMK
      IF(MK-N-1)150,140,150
  140 BJ=BMK
  150 JT=-JT
      S=1+JT
  160 ALPHA=ALPHA+BMK*S
      BMK=2.*FM1/X-FM
      IF(N)180,170,180
  170 BJ=BMK
  180 ALPHA=ALPHA+BMK
      BJ=BJ/ALPHA
          IF(DABS(BJ-BPREV)-DABS(D*BJ))200,200,190
  190 BPREV=BJ
      IER=3
  200 RETURN
      END
```

APPENDIX B

ROOTS OF THE SHIFTED LEGENDRE POLYNOMIALS AND CORRESPONDING WEIGHTS

The roots, x_i, of the shifted Legendre polynomial $P_N^*(x)$ are computed from the roots of Legendre polynomial $P_N(r)$ using the formula

$$x_i = \frac{1+r_i}{2} \quad .$$

The coefficients (or weights) of the shifted Legendre polynomial $P_N^*(x)$ in the interval (0,1) are then

$$A_i^* = \frac{A_i}{2} \quad ,$$

where A_i's are the weights of the Legendre polynomial $P_N(r)$ in the interval (-1,1) and were obtained from Stroud and Secrest [20].

ROOTS AND WEIGHTS

ROOTS		WEIGHTS
	N = 2	
0.7886751345948126OD 00		0.5000000000000000D 00
0.2113248654051869OD 00		0.5000000000000000D 00
	N = 3	
0.8872983346207412OD 00		0.2777777777777777OD 00
0.5000000000000000D 00		0.4444444444444442OD 00
0.1127016653792583OD 00		0.2777777777777777OD 00
	N = 4	
0.9305681557970260OD 00		0.1739274225687266OD 00
0.6699905217924278OD 00		0.3260725774312727OD 00
0.3300094782075718OD 00		0.3260725774312727OD 00
0.6943184420297380OD-01		0.1739274225687266OD 00
	N = 5	
0.9530899229693315OD 00		0.1184634425280944OD 00
0.7692346550528413OD 00		0.2393143352496829OD 00
0.5000000000000000D 00		0.2844444444444443OD 00
0.2307653449471583OD 00		0.2393143352496829OD 00
0.4691007703066814OD-01		0.1184634425280944OD 00
	N = 6	
0.9662347571015757OD 00		0.8566224618958480OD-01
0.8306046932331319OD 00		0.1803807865240692OD 00
0.6193095930415984OD 00		0.2339569672863453OD 00
0.3806904069584014OD 00		0.2339569672863453OD 00
0.1693953067668674OD 00		0.1803807865240692OD 00
0.3376524289842404OD-01		0.8566224618958480OD-01
	N = 7	
0.9745539561713792OD 00		0.6474248308443450OD-01
0.8707655927996969OD 00		0.1398526957446380OD 00
0.7029225756886983OD 00		0.1909150252525591OD 00
0.5000000000000000D 00		0.2089795918367343OD 00
0.2970774243113015OD 00		0.1909150252525591OD 00
0.1292344072003029OD 00		0.1398526957446380OD 00
0.2544604382862077OD-01		0.6474248308443450OD-01

ROOTS AND WEIGHTS (continued)

ROOTS	WEIGHTS
N = 8	
0.98014492824876800D 00	0.50614268145188080D-01
0.89833323870681300D 00	0.11119051722668690D 00
0.76276620495816400D 00	0.15685332293894350D 00
0.59171732124782460D 00	0.18134189168918070D 00
0.40828267875217520D 00	0.18134189168918070D 00
0.23723379504183530D 00	0.15685332293894350D 00
0.10166676129318650D 00	0.11119051722668690D 00
0.19855071751231980D-01	0.50614268145188080D-01
N = 9	
0.98408011975381270D 00	0.40637194180787150D-01
0.91801555366331740D 00	0.90324080347428300D-01
0.80668571635029520D 00	0.13030534820146730D 00
0.66212671170190430D 00	0.15617353852000140D 00
0.50000000000000000D 00	0.16511967750062960D 00
0.33787328829809550D 00	0.15617353852000140D 00
0.19331428364970460D 00	0.13030534820146730D 00
0.81984446336681900D-01	0.90324080347428300D-01
0.15919880246187050D-01	0.40637194180787150D-01
N = 10	
0.98695326425858540D 00	0.33335672154344030D-01
0.93253168334449190D 00	0.74725674575290000D-01
0.83970478414951200D 00	0.10954318125799080D 00
0.71669769706462350D 00	0.13463335965499780D 00
0.57443716949081550D 00	0.14776211235737600D 00
0.42556283050918410D 00	0.14776211235737600D 00
0.28330230293537630D 00	0.13463335965499780D 00
0.16029521585048780D 00	0.10954318125799080D 00
0.67468316655507700D-01	0.74725674575290000D-01
0.13046735741414190D-01	0.33335672154344030D-01
N = 11	
0.98911432907302820D 00	0.27834283558086790D-01
0.94353129988404730D 00	0.62790184732451900D-01
0.86507600278702430D 00	0.93145105463866800D-01
0.75954806460340560D 00	0.11659688229599510D 00
0.63477157797617220D 00	0.13140227225512290D 00
0.50000000000000000D 00	0.13646254338894990D 00
0.36522842202382730D 00	0.13140227225512290D 00
0.24045193539659390D 00	0.11659688229599510D 00
0.13492399721297520D 00	0.93145105463866800D-01
0.56468700115952460D-01	0.62790184732451900D-01
0.10885670926971520D-01	0.27834283558086790D-01

ROOTS AND WEIGHTS (continued)

N = 12

ROOTS	WEIGHTS
0.99078031712335930D 00	0.23587668193255860D-01
0.95205862818523700D 00	0.53469662997659180D-01
0.88495133709715200D 00	0.80039164271672700D-01
0.79365897714330840D 00	0.10158371336153270D 00
0.68391574949908970D 00	0.11674626826917710D 00
0.56261670425573420D 00	0.12457352290670110D 00
0.43738329574426560D 00	0.12457352290670110D 00
0.31608425050090960D 00	0.11674626826917710D 00
0.20634102285669110D 00	0.10158371336153270D 00
0.11504866290284750D 00	0.80039164271672700D-01
0.47941371814762610D-01	0.53469662997659180D-01
0.92196828766404250D-02	0.23587668193255860D-01

N = 13

ROOTS	WEIGHTS
0.99209152735929360D 00	0.20242002382657900D-01
0.95879919961148860D 00	0.46060749918864170D-01
0.90078904536665450D 00	0.69436755109893600D-01
0.82117466972016980D 00	0.89072990380972600D-01
0.72424637551822310D 00	0.10390802376844400D 00
0.61522915797756730D 00	0.11314159013144830D 00
0.50000000000000000D 00	0.11627577661543650D 00
0.38477084202243230D 00	0.11314159013144830D 00
0.27575362448177640D 00	0.10390802376844400D 00
0.17882533027982970D 00	0.89072990380972600D-01
0.99210954633345100D-01	0.69436755109893600D-01
0.41200800388511100D-01	0.46060749918864170D-01
0.79084726407060300D-02	0.20242002382657900D-01

N = 14

ROOTS	WEIGHTS
0.99314190434840570D 00	0.17559730165875880D-01
0.96421744183178630D 00	0.40079043579880060D-01
0.91360065753488210D 00	0.60759285343951540D-01
0.84364645240584240D 00	0.78601583579096500D-01
0.75762431817907670D 00	0.92769198738968500D-01
0.65955618446394460D 00	0.10259923186064770D 00
0.55402747435367150D 00	0.10763192673157860D 00
0.44597252564632830D 00	0.10763192673157860D 00
0.34044381553605520D 00	0.10259923186064770D 00
0.24237568182092280D 00	0.92769198738968500D-01
0.15635354759415710D 00	0.78601583579096500D-01
0.86399342465117200D-01	0.60759285343951540D-01
0.35782558168213250D-01	0.40079043579880060D-01
0.68580956515938870D-02	0.17559730165875880D-01

N = 15

ROOTS	WEIGHTS
0.993996259010242200D 00	0.153766209980585900D-01
0.968636696200352500D 00	0.351830237440540200D-01
0.924103291705213200D 00	0.535796102335859300D-01
0.862208865680084600D 00	0.697853389630769000D-01
0.785486086304269200D 00	0.831346029084967000D-01
0.697075673538781300D 00	0.930805000077808000D-01
0.600597046998717200D 00	0.992157426635556000D-01
0.500000000000000000D 00	0.101289120962780200D 00
0.399402953001282600D 00	0.992157426635556000D-01
0.302924326461218000D 00	0.930805000077808000D-01
0.214513913695730600D 00	0.831346029084967000D-01
0.137791134319914700D 00	0.697853389630769000D-01
0.758967082947861000D-01	0.535796102335859300D-01
0.313633037996470900D-01	0.351830237440540200D-01
0.600374098975741100D-02	0.153766209980585900D-01

APPENDIX C

Legendre Coefficients

This table gives the coefficients of the first fifteen shifted Legendre polynomials, $P_N^*(x)$, in order of lowest to highest power such that

$$P_N^*(x) = \sum_{K=0}^{\infty} C_{NK} x^K ,$$

where $P_N^*(x) = P_N(1-2x)$. The recurrence relation for these coefficients is

$$(N-1)C_{NK} = (2N-3)C_{N-1,K} - (N-2)C_{N-2,K} - (4N-6)C_{N-1,K-1} .$$

All of these coefficients are integers but due to machine conversion from hexadecimal to decimal, noninteger numbers may be printed in this table. In other words

-0.13999999999999990D 03 represents the integer -14. The coefficients are printed in order of lowest to highest power of x. For example

$$P_3^*(x) = 1-6x-6x^2.$$

LEGENDRE COEFFICIENTS

N = 1
0.99999999999999900D 00
-0.20000000000000000D 01

N = 2
0.99999999999999900D 00
-0.59999999999999990D 01
0.59999999999999990D 01

N = 3
0.99999999999999900D 00
-0.11999999999999990D 02
0.29999999999999980D 02
-0.20000000000000000D 02

N = 4
0.99999999999999900D 00
-0.20000000000000000D 02
0.89999999999999990D 02
-0.13999999999999990D 03
0.70000000000000000D 02

N = 5
0.99999999999999900D 00
-0.29999999999999980D 02
0.21000000000000000D 03
-0.55999999999999980D 03
0.62999999999999990D 03
-0.25199999999999980D 03

N = 6
0.99999999999999900D 00
-0.41999999999999990D 02
0.41999999999999990D 03
-0.16799999999999990D 04
0.31499999999999990D 04
-0.27719999999999990D 04
0.92399999999999990D 03

N = 7
0.99999999999999900D 00
-0.55999999999999980D 02
0.75600000000000000D 03
-0.41999999999999990D 04
0.11549999999999990D 05
-0.16631999999999990D 05
0.12011999999999990D 05
-0.34319999999999990D 04

N = 8
0.99999999999999900D 00
-0.72000000000000000D 02
0.12599999999999990D 04
-0.92399999999999990D 04
0.34649999999999980D 05
-0.72071999999999980D 05
0.84083999999999980D 05
-0.51479999999999990D 05
0.12869999999999980D 05

N = 9
0.999999999999999900D 00
-0.899999999999999990D 02
0.198000000000000000D 04
-0.184799999999999980D 05
0.900899999999999980D 05
-0.252251999999999990D 06
0.420419999999999980D 06
-0.411839999999999970D 06
0.218789999999999980D 06
-0.486199999999999970D 05

N = 10
0.999999999999999900D 00
-0.109999999999999990D 03
0.296999999999999990D 04
-0.343199999999999970D 05
0.210209999999999980D 06
-0.756755999999999970D 06
0.168167999999999990D 07
-0.233375999999999980D 07
0.196910999999999980D 07
-0.923779999999999980D 06
0.184755999999999990D 06

N - 11

0.99999999999999900D	00
-0.13199999999999990D	03
0.42899999999999980D	04
-0.60059999999999980D	05
0.45044999999999980D	06
-0.20180159999999980D	07
0.57177119999999990D	07
-0.10501919999999990D	08
0.12471029999999980D	08
-0.92377999999999980D	07
0.38798759999999970D	07
-0.70543199999999980D	06

N = 12

0.99999999999999900D	00
-0.15599999999999990D	03
0.60059999999999980D	04
-0.10009999999999980D	06
0.90089999999999980D	06
-0.49008959999999990D	07
0.17153135999999990D	08
-0.39907295999999980D	08
0.62355149999999980D	08
-0.64664599999999980D	08
0.42678635999999990D	08
-0.16224935999999990D	08
0.27041560000000000D	07

N = 13

0.99999999999999900D	00
-0.18199999999999990D	03
0.81899999999999990D	04
-0.16015999999999980D	06
0.17016999999999990D	07
-0.11027015999999980D	08
0.46558511999999980D	08
-0.13302431999999990D	09
0.26189162999999980D	09
-0.35565529999999980D	09
0.32720287599999990D	09
-0.19469923199999980D	09
0.67603899999999980D	08
-0.10400599999999980D	08

N = 14

0.99999999999999900D	00
-0.21000000000000000D	03
0.10920000000000000D	05
-0.24752000000000000D	06
0.30630600000000000D	07
-0.23279255999999980D	08
0.11639627999999980D	09
-0.39907295999999980D	09
0.96026930999999990D	09
-0.16360143799999990D	10
0.19632172559999980D	10
-0.16224935999999990D	10
0.87885069999999990D	09
-0.28081619999999980D	09
0.40116599999999990D	08

N = 15

0.99999999999999900D	00
-0.23999999999999980D	03
0.14279999999999980D	05
-0.37127999999999980D	06
0.52907399999999980D	07
-0.46558511999999980D	08
0.27159131999999990D	09
-0.10974506399999990D	10
0.31551705899999980D	10
-0.65440575199999970D	10
0.98160862799999960D	10
-0.10546208399999980D	11
0.79096562999999960D	10
-0.39314267999999970D	10
0.11633813999999980D	10
-0.15511751999999990D	09

APPENDIX D

Negatives of Logarithms of Roots of
Shifted Legendre Polynomials

The negative logarithms computed in these tables are

$$t_i = -\log x_i \quad ,$$

where

$$P_N^*(x_i) = 0 \quad , \qquad i = 1, 2, \ldots , N; \quad N = 3, 4, \ldots , 15.$$

It can be readily observed from these tables that the times or t values interlace for different N, making possible the use of more t values for extrapolation purposes.

NEGATIVES OF LOGARITHMS OF ROOTS OF SHIFTED LEGENDRE POLYNOMIALS

T = -LOG(x)

N = 2

0.23740078615161920D 00
0.15543586830764310D 01

N = 3

0.11957401204924280D 00
0.69314718055994510D 00
0.21830110809448010D 01

N = 4

0.71959959158763400D-01
0.40049171327565440D 00
0.11086339030929460D 01
0.26674096665219890D 01

N = 5

0.48046021994112870D-01
0.26235921291155620D 00
0.69314718055994510D 00
0.14663539074975080D 01
0.30595227645482920D 01

N = 6

0.34348454520252990D-01
0.18560129738833380D 00
0.47914998102148920D 00
0.96576881433932520D 00
0.17755202021841730D 01
0.33883233221881020D 01

N = 7

0.25775393510447510D-01
0.13838246257543370D 00
0.35250852739216200D 00
0.69314718055994510D 00
0.12137624862357850D 01
0.20461274134878730D 01
0.36711949968461940D 01

N = 8

0.20054832283440630D-01
0.10721418964279870D 00
0.27080371017260680D 00
0.52472625602874090D 00
0.89579550440199850D 00
0.14387091471258830D 01
0.22860548602906120D 01
0.39192958006440780D 01

N = 9

0.16047962730479190D-01
0.85540945516813900D-01
0.21482113346806830D 00
0.41229833395649940D 00
0.69314718055994510D 00
0.10850843404787260D 01
0.16434380018108130D 01
0.25012257285301320D 01
0.41401866208186200D 01

N = 10

0.13132591978187970D-01
0.69852151264428800D-01
0.17470489635796030D 00
0.33310114923617220D 00
0.55436455322540930D 00
0.85434267909849090D 00
0.12612407432151420D 01
0.18307380647871180D 01
0.26960971741796480D 01
0.43392173112598770D 01

N = 11

0.10945353360154720D-01
0.58125740495859540D-01
0.14493791141644620D 00
0.27503167441491130D 00
0.45449006452666760D 00
0.69314718055994510D 00
0.10072323073755050D 01
0.14252350622303700D 01
0.20030436426860710D 01
0.28740687745821190D 01
0.45203079484393840D 01

NEGATIVES OF LOGARITHMS OF ROOTS OF SHIFTED LEGENDRE POLYNOMIALS (continued)

N = 12

0.92624472047904000D-02
0.49128661858812220D-01
0.12222262181691270D 00
0.23110140981597780D 00
0.37992054219297830D 00
0.57515669219094700D 00
0.82694536135380250D 00
0.11517464854216140D 01
0.15782250281801380D 01
0.21624000844470620D 01
0.30377764351420280D 01
0.46864146371634650D 01

N = 13

0.79399104703031280D-02
0.42073611203765560D-01
0.10448418267570260D 00
0.19701943975857120D 00
0.32262364680743880D 00
0.48576046598152780D 00
0.69314718055994510D 00
0.95510733743158590D 00
0.12882474768486800D 01
0.17213457580450050D 01
0.23105068410166020D 01
0.31892975959097460D 01
0.48398206080462550D 01

N = 14

0.68817204656915940D-02
0.36438447740720890D-01
0.90361720362083900D-01
0.17002176740104690D 00
0.27756763865621490D 00
0.41618811793798540D 00
0.59054100076837180D 00
0.80749793054938170D 00
0.10775051727753410D 01
0.14172663523405600D 01
0.18556355052547450D 01
0.24487752135585180D 01
0.33302947063021520D 01
0.49823254775913270D 01

N = 15

0.60218359038299730D-02
0.31865663915715830D-01
0.78931426028461700D-01
0.14825773409602940D 00
0.24145253461422690D 00
0.36086130375801570D 00
0.50983104019704670D 00
0.69314718055994510D 00
0.91778446443415710D 00
0.11942722523176470D 01
0.15393806770033880D 01
0.19820162597733620D 01
0.25783819644961590D 01
0.34621167382948760D 01
0.51153725057554800D 01

APPENDIX E

Legendre Derivatives

The derivatives of the shifted Legendre polynomials tabulated in this table are evaluated at the roots and listed in order of smallest to largest root. For example,

$$P_5^{*'}\ (0.5000) = -3.75.$$

Recalling that the recurrence relation for the derivative of the Legendre polynomial is

$$(x^2-1)\ P_N'\ (x) = N\ xP_N(x)-P_{N-1}(x) \quad ,$$

and the relationships between shifted Legendre polynomials and Legendre polynomials are

$$P_N^*(x) = P_N(1-2x) \quad ,$$

and

$$\frac{d}{dx}\ P_N^*(x) = -2P_N'\ (1-2x) \quad ,$$

we can determine the derivative of the shifted polynomial from the formula

$$-2x(x-1)\frac{d}{dx}\ P_N^*(x) = N[(1-2x)P_N^*(x)-P_{N-1}^*(x)] \quad .$$

LEGENDRE DERIVATIVES

N = 3

-0.59999999999999560D 01
0.29999999999999980D 01
-0.59999999999999990D 01

N = 4

0.94332756589370900D 01
-0.37243057339114480D 01
0.37243057339114550D 01
-0.94332756589371720D 01

N = 5

-0.13740666961489280D 02
0.48517780726008250D 01
-0.37499999999999980D 01
0.48517780726008380D 01
-0.13740666961489610D 02

N = 6

0.18915986873541660D 02
-0.62770701859863770D 01
0.42578674578469820D 01
-0.42578674578469820D 01
0.62770701859864400D 01
-0.18915986873542170D 02

N = 7

-0.24956992249836780D 02
0.79712213924980360D 01
-0.50083119828975910D 01
0.43749999999999980D 01
-0.50083119828975930D 01
0.79712213924980850D 01
-0.24956992249837180D 02

N = 8

0.31862704825655890D 02
-0.99233405513432490D 01
0.59356616715355440D 01
-0.47776387675098840D 01
0.47776387675098840D 01
-0.59356616715355900D 01
0.99233405513433910D 01
-0.31862704825656630D 02

N = 9

-0.39632641191048770D 02
0.12128510598590920D 02
-0.70151138493662570D 01
0.53499385187671650D 01
-0.49218749999999980D 01
0.53499385187671720D 01
-0.70151138493662680D 01
0.12128510598591260D 02
-0.39632641191052960D 02

N = 10

0.48266539729458070D 02
-0.14584224552352420D 02
0.82353816679841920D 01
-0.60482620784350310D 01
0.52615698870016580D 01
-0.52615698870016690D 01
0.60482620784350360D 01
-0.82353816679842400D 01
0.14584224552352820D 02
-0.48266539729465570D 02

N = 11

-0.57764248669799700D 02
0.17289087349651640D 02
-0.95906545604470890D 01
0.68527480037784350D 01
-0.57293788755183090D 01
0.54140624999999990D 01
-0.57293788755183200D 01
0.68527480037784640D 01
-0.95906545604471760D 01
0.17289087349652200D 02
-0.57764248669807150D 02

LEGENDRE DERIVATIVES (continued)

N = 12

```
 0.681256749893980100D 02
-0.202422694445930800D 02
 0.110776654369707300D 02
-0.775315027696226200D 01
 0.629471055940404600D 01
-0.571149392612676300D 01
 0.571149392612676500D 01
-0.629471055940408000D 01
 0.775315027696230600D 01
-0.110776654369708400D 02
 0.202422694445950600D 02
-0.681256749894093100D 02
```

N = 13

```
-0.793507590688193900D 02
 0.234432511315910500D 02
-0.126944511594655700D 02
 0.874368882746129500D 01
-0.694179329378264300D 01
 0.611039560984343500D 01
-0.586523437499999900D 01
 0.611039560984344600D 01
-0.694179329378267400D 01
 0.874368882746138900D 01
-0.126944511594659500D 02
 0.234432511315919100D 02
-0.793507590688431000D 02
```

N = 14

```
 0.914394612546795900D 02
-0.268916927619570000D 02
 0.144397691803535000D 02
-0.982087627944362600D 01
 0.766172609937013400D 01
-0.658839034758065500D 01
 0.613210721199302500D 01
-0.613210721199303000D 01
 0.658839034758066100D 01
-0.766172609937018900D 01
 0.982087627944376900D 01
-0.144397691803540500D 02
 0.268916927619592800D 02
-0.914394612547139600D 02
```

N = 15

```
-0.104391754335494400D 03
 0.305873644457662590D 02
-0.163127994164558200D 02
 0.109824952496234200D 02
-0.844913219511648200D 01
 0.713285647640509500D 01
-0.648205145519371800D 01
 0.628417968749999800D 01
-0.648205145519371800D 01
 0.713285847640514600D 01
-0.844913219511653100D 01
 0.109824952496236700D 02
-0.163127994164567300D 02
 0.305873644457665940D 02
-0.104391754335544600D 03
```

APPENDIX F

Coefficients of the Polynomials

$P_N^*(x)/(x-x_i)$

The coefficients of the polynomials $P_N^*(x)/(x-x_i)$ are tabulated in order of smallest to largest root, x_i, and lowest to highest power. For example,

$$P_5^*(x)/(x-0.5000) = -1.05x^0+30.4x^1-188.5x^2+389.8x^3-252x^4.$$

These are the coefficients of the polynomials

$$\frac{P_N^*(x)}{x-x_i} = \sum_{K=0}^{N-1} b_{ik}x^K \quad ,$$

where x_i is a root of the shifted Legendre polynomial $P_N^*(x)$, and the coefficients were obtained by synthetic division

$$\begin{aligned}
b_{i,N-1} &= C_{NN} \\
b_{i,N-2} &= C_{N,N-1}+x_i b_{i,N-1} \\
&\vdots \\
b_{i,k-1} &= C_{N,k}=x_i b_{ik} \\
&\vdots \\
b_{i,0} &= C_{N1}+x_i b_{i1} \quad , \quad i = 1,2, \ldots, N.
\end{aligned}$$

The Legendre coefficients C_{NM} are defined by

$$P_N^*(x) = \sum_{M=0}^{N} C_{NM}x^M.$$

COEFFICIENTS OF THE POLYNOMIALS $P_n^*(x)/(x-x_i)$

N = 3

I = 1

-0.11270166537925780D 01
0.12254033307585160D 02
-0.20000000000000000D 02

I = 2

-0.20000000000000000D 01
0.20000000000000000D 02
-0.20000000000000000D 02

I = 3

-0.88729833462074170D 01
0.27745966692414800D 02
-0.20000000000000000D 02

N = 4

I = 1

-0.10746123148212680D 01
0.20337454669259780D 02
-0.74860229094208130D 02
0.70000000000000000D 02

I = 2

-0.14925584280277500D 01
0.27623437899478340D 02
-0.93100663474529990D 02
0.70000000000000000D 02

I = 3

-0.30302159969205870D 01
0.51422110950418290D 02
-0.11689933652546980D 03
0.70000000000000000D 02

I = 4

-0.14402613260230380D 02
0.80616996480843440D 02
-0.13513977090579180D 03
0.70000000000000000D 02

N = 5

I = 1

-0.10492189413613050D 01
0.30375707853927490D 02
-0.18846520964827400D 03
0.38982133941172820D 03
-0.25199999999999980D 03

I = 2

-0.12999934329938890D 01
0.37309820063988910D 02
-0.22449609985934440D 03
0.43615286692668340D 03
-0.25199999999999980D 03

I = 3

-0.20000000000000000D 01
0.55999999999999980D 02
-0.30799999999999980D 03
0.50400000000000000D 03
-0.25199999999999980D 03

I = 4

-0.43334063016653720D 01
0.11122377887464790D 03
-0.42803749907929230D 03
0.57184713307331550D 03
-0.25199999999999980D 03

I = 5

-0.21317381323979400D 02
0.18509069320743520D 03
-0.53100119141308810D 03
0.61817866058827150D 03
-0.25199999999999980D 03

COEFFICIENTS OF THE POLYNOMIALS

$P_n^*(x)/(x-x_i)$ (continued)

N = 6

I = 1

-0.1034945175228186OD 01
0.4239658584385339OD 02
-0.3907988316305726OD 03
0.1334252529102404OD 04
-0.1879199084438141OD 04
0.9239999999999999OD 03

I = 2

-0.1203942149793895OD 01
0.4911609359129345OD 02
-0.4465227676056573OD 03
0.1485035230890672OD 04
-0.2004521263452585OD 04
0.9239999999999999OD 03

I = 3

-0.1614701291947884OD 01
0.6521019399959378OD 02
-0.5728795581188022OD 03
0.1787668807847502OD 04
-0.2199757936029562OD 04
0.9239999999999999OD 03

I = 4

-0.2626806406785209OD 01
0.1034257571862511OD 03
-0.8315792492463332OD 03
0.2228637063729251OD 04
-0.2420242063970435OD 04
0.9239999999999999OD 03

I = 5

-0.5903351273930293OD 01
0.2130911972416576OD 03
-0.1221455344350848OD 04
0.2706950177080329OD 04
-0.2615478736547412OD 04
0.9239999999999999OD 03

(N = 6)

I = 6

-0.2961625370231449OD 02
0.3667601721373501OD 03
-0.1576764249047779OD 04
0.3057456191349828OD 04
-0.2740800915561854OD 04
0.9239999999999999O 03

N = 7

I = 1

-0.1026110451522581OD 01
0.5640928262653324OD 02
-0.7178573468849998OD 03
0.3573062970053400OD 04
-0.8185218457564622OD 04
0.8667330822419825OD 04
-0.3431999999999999OD 04

I = 2

-0.1148414691932091OD 01
0.6299236644354351OD 02
-0.7958601479972229OD 03
0.3909364219431020OD 04
-0.8774618386106303OD 04
0.9023532485511430OD 04
-0.3431999999999999OD 04

I = 3

-0.1422631787035966OD 01
0.7764349887253381OD 02
-0.9650515214464930OD 03
0.4602140535014141OD 04
-0.9884245726748193OD 04
0.9599569720236380OD 04
-0.3431999999999999OD 04

COEFFICIENTS OF THE POLYNOMIALS $P_n^{*}(x)/(x-x_i)$ (continued)

(N = 7)

I = 4

-0.20000000000000000D 01
0.10799999999999990D 02
-0.12959999999999990D 03
0.58080000000000000D 04
-0.11483999999999980D 05
0.10295999999999990D 05
-0.34319999999999990D 04

I = 5

-0.33661258586654900D 01
0.17717224478888880D 03
-0.19484070745293940D 04
0.75791451696147850D 04
-0.13366397125566240D 05
0.10992430279763600D 05
-0.34319999999999990D 04

I = 6

-0.77378774094586310D 01
0.37344638812587250D 03
-0.29601529512276080D 04
0.95937844698797980D 04
-0.15136959956549090D 05
0.11568467514468540D 05
-0.34319999999999990D 04

I = 7

-0.39298839801385420D 02
0.65633621914262670D 03
-0.39166709579142760D 04
0.11134502636006810D 05
-0.16328564345465460D 05
0.11924669177580150D 05
-0.34319999999999990D 04

N = 8

I = 1

-0.10202572815328950D 01
0.72417599349605480D 02
-0.12116395916798290D 04
0.81909931653316720D 04
-0.26994994385108750D 05
0.45990143208138260D 05
-0.38865534773438350D 05
0.12869999999999980D 05

I = 2

-0.11131726590019590D 01
0.78909278079304370D 02
-0.13147578994415500D 04
0.88221628223031720D 04
-0.28750842187334650D 05
0.48223928433314890D 05
-0.39918451217843300D 05
0.12869999999999980D 05

I = 3

-0.13110177056897280D 01
0.92674507384851140D 02
-0.15303843891185150D 04
0.10107442569908190D 05
-0.32175727333695730D 05
0.52304719856449930D 05
-0.41663198942188390D 05
0.12869999999999980D 05

I = 4

-0.16899961587937520D 01
0.11882363641634660D 03
-0.19285836709615300D 04
0.12356265511410070D 05
-0.37676325650863890D 05
0.58128557529128660D 05
-0.43864598075540460D 05
0.12869999999999980D 05

COEFFICIENTS OF THE POLYNOMIALS

$P_n^*(x)/(x-x_i)$ (continued)

(N = 8)

I = 5

-0.24492834304319100D 01
0.17034941766849940D 03
-0.26688631162648720D 04
0.16094576688431480D 05
-0.45447490861672100D 05
0.65210969075885700D 05
-0.46225401924459470D 05
0.12869999999999980D 05

I = 6

-0.42152510346328980D 01
0.28572973320859910D 03
-0.41067937501045800D 04
0.21637752955856010D 05
-0.54849887815727500D 05
0.72595526203319190D 05
-0.48426601057811540D 05
0.12869999999999980D 05

I = 7

-0.98360564188348000D 01
0.61144805628160780D 03
-0.63791935089591110D 04
0.28139054049247060D 05
-0.64042031711590040D 05
0.78983221126254860D 05
-0.50171548762156660D 05
0.12869999999999980D 05

I = 8

-0.50364965311090020D 02
0.10896477716111730D 04
-0.85797840734699980D 04
0.33251752237512180D 05
-0.70422700054007100D 05
0.83066934567508040D 05
-0.51224465226561610D 05
0.12869999999999980D 05

N = 9

I = 1

-0.10161774228702810D 01
0.90423351504540900D 02
-0.19201451290045090D 04
0.16827750646093990D 05
-0.74447443742928090D 05
0.18068097575383680D 06
-0.24361738382250640D 06
0.17094402457756930D 06
-0.48619999999999970D 05

I = 2

-0.10893061626269460D 01
0.96850966721181560D 02
-0.20513258471102790D 04
0.17895855998660920D 05
-0.78641525966798870D 05
0.18911495926230540D 06
-0.25196200632405130D 06
0.17415608378088930D 06
-0.48619999999999970D 05

I = 3

-0.12396401469971320D 01
0.11003090553602840D 03
-0.23180887631484420D 04
0.20034954021466020D 05
-0.86843047494983990D 05
0.20504757819858140D 06
-0.26698430062184840D 06
0.17956894047104830D 06
-0.48619999999999970D 05

I = 4

-0.15102849384090920D 01
0.13364468386140230D 03
-0.27885226249108700D 04
0.23698601940943240D 05
-0.10026992852834280D 06
0.22953623345146140D 06
-0.28828887760455710D 06
0.18659739927705310D 06
-0.48619999999999970D 05

COEFFICIENTS OF THE POLYNOMIALS

$P_n^*(x)/(x-x_i)$ (continued)

(N = 9)

I = 5

-0.2000000000000000D 01
0.1759999999999990D 03
-0.3607999999999980D 04
0.2974399999999990D 05
-0.1206919999999980D 06
0.2631199999999980D 06
-0.3145999999999980D 06
0.1944799999999970D 06
-0.4861999999999970D 05

I = 6

-0.2959689429836032OD 01
0.2576122871642072OD 03
-0.5097732707760485OD 04
0.3960735505208901OD 05
-0.1494129506425248OD 06
0.3043716473577602OD 06
-0.3434670826651631OD 06
0.2023626007229464OD 06
-0.4861999999999997OD 05

I = 7

-0.5172923495979373OD 01
0.4388039771428975OD 03
-0.7972489118547627OD 04
0.5435454992292499OD 05
-0.1848567493431028OD 06
0.3486304756404883OD 06
-0.3713617173245076OD 06
0.2093910595289512OD 06
-0.4861999999999997OD 05

I = 8

-0.1219743554641257OD 02
0.9489917652682424OD 03
-0.1257565649096122OD 05
0.7201784939537987OD 05
-0.2204338921849132OD 06
0.3880993192833246OD 06
-0.3942294198578249OD 06
0.2148039162191102OD 06
-0.4861999999999997OD 05

(N = 9)

I = 9

-0.6281454285684793OD 02
0.1707642062801520OD 04
-0.1710803931855649OD 05
0.8617908302244170OD 05
-0.2456624620964038OD 06
0.4139188110522400OD 06
-0.4083692117795190OD 06
0.2180159754224302OD 06
-0.4861999999999997OD 05

(N = 10)

I = 1

-0.1013219203018738OD 01
0.1104274991975975OD 03
-0.2897373770733262OD 04
0.3163800831022316OD 05
-0.1807299272917164OD 06
0.5836406784074042OD 06
-0.1112554526497724OD 07
0.1237348836795895OD 07
-0.7414344627086404OD 06
0.1847559999999999OD 06

I = 2

-0.1072349623863344OD 01
0.1168085249237551OD 03
-0.3059618805490203OD 04
0.3352205801994366OD 05
-0.1894712481471632OD 06
0.6083275903487693OD 06
-0.1151009052906049OD 07
0.1268322533398598OD 07
-0.7514891763120048OD 06
0.1847559999999999OD 06

COEFFICIENTS OF THE POLYNOMIALS

$P_n^*(x)/(x-x_i)$ (continued)

(N = 10)

I = 3

-0.11908947273082650D	01
0.12958018976025950D	03
-0.33826409755621850D	04
0.36843137741286600D	05
-0.20646168216643710D	06
0.65534260161554700D	06
-0.12222597962496650D	07
0.13236797321288390D	07
-0.76863950289967240D	06
0.18475599999999990D	06

I = 4

-0.13952884236969410D	01
0.15153489682067490D	03
-0.39325717310421160D	04
0.42399226889406180D	05
-0.23414442909178530D	06
0.72919387497500370D	06
-0.13289928639733150D	07
0.14019399534028090D	07
-0.79136580028112810D	06
0.18475599999999990D	06

I = 5

-0.17408344256148520D	01
0.18846128231978200D	03
-0.48421983559068590D	04
0.51315971893362030D	05
-0.27660819415199490D	06
0.83585782980166610D	06
-0.14724363518260360D	07
0.14994218583338540D	07
-0.81764928631355450D	06
0.18475599999999990D	06

(N = 10)

I = 6

-0.23498292809155160D	01
0.25295952325131730D	03
-0.63845812696981860D	04
0.65643464907114140D	05
-0.33970667720184200D	06
0.97999471029732480D	06
-0.16488406397314120D	07
0.16094435678254130D	07
-0.84515471368644480D	06
0.18475599999999990D	06

I = 7

-0.35297983448739930D	01
0.37581834158055780D	03
-0.91569381241888310D	04
0.88820534161174900D	05
-0.42848033560290120D	06
0.11587468968509620D	07
-0.18458484019747700D	07
0.17222295511537820D	07
-0.87143819971887110D	06
0.18475599999999990D	06

I = 8

-0.62384893690947910D	01
0.64731508099210330D	03
-0.14490045174987220D	05
0.12370896236546900D	06
-0.53963580369067550D	06
0.13545020364415510D	07
-0.20410962474613890D	07
0.18257797089314560D	07
-0.89416449710032710D	06
0.18475599999999990D	06

(N = 10)

I = 9

-0.14821771900816640D 02
0.14107099868099880D 04
-0.23111440902723630D 05
0.16613070627665280D 06
-0.65333323711654740D 06
0.15329086008107720D 07
-0.22050557441480030D 07
0.19076251229025560D 07
-0.91131482368799460D 06
0.18475599999999990D 06

I = 10

-0.76647524700428040D 02
0.25563846743443450D 04
-0.31702590889667050D 05
0.20061792943536710D 06
-0.73520846553093450D 06
0.16515651804509890D 07
-0.23082263752316030D 07
0.19570891351267680D 07
-0.92136953729135910D 06
0.18475599999999990D 06

N = 11

I = 1

-0.10110054727960090D 01
0.13243059035446710D 03
-0.42033254270432940D 04
0.56471403690313600D 05
-0.39831451706792320D 06
0.16375270636813190D 07
-0.41250893009936650D 07
0.64470107363448310D 07
-0.60903164443090320D 07
0.31821231006133540D 07
-0.70543199999999980D 06

(N = 11)

I = 2

-0.10598482530719400D 01
0.13877669109972230D 03
-0.43996667724858990D 04
0.58991506942434560D 05
-0.41488660005838970D 06
0.16990738941449220D 07
-0.42591465766413190D 07
0.66163925077269420D 07
-0.62050273191705860D 07
0.32142788280601930D 07
-0.70543199999999980D 06

I = 3

-0.11559677957135300D 01
0.15125148747941770D 03
-0.47842599947134420D 04
0.63896975326103250D 05
-0.44684284782901690D 06
0.18162255652787900D 07
-0.45099926736514970D 07
0.69264750230549010D 07
-0.64093269944861990D 07
0.32696237052019400D 07
-0.70543199999999980D 06

I = 4

-0.13165723758385340D 01
0.17205419079357020D 03
-0.54215736977179900D 04
0.71935442730423050D 05
-0.49834180996460860D 06
0.20007610589184740D 07
-0.48936349314803500D 07
0.73837131971996860D 07
-0.66978207698503270D 07
0.33440664896906870D 07
-0.70543199999999980D 06

(N = 11)

I = 5

-0.15753698412222740D 01
0.20546702890291260D 03
-0.64346500580881540D 04
0.84479759022740410D 05
-0.57653848041538610D 06
0.22708602111336490D 07
-0.54300663552956530D 07
0.79900452708907000D 07
-0.70592082011546810D 07
0.34320878162051090D 07
-0.70543199999999980D 06

I = 6

-0.20000000000000000D 01
0.25999999999999980D 03
-0.80599999999999980D 04
0.10399999999999990D 06
-0.69289999999999980D 06
0.26502319999999990D 07
-0.61349599999999980D 07
0.87339199999999990D 07
-0.74742199999999980D 07
0.35271599999999980D 07
-0.70543199999999980D 06

I = 7

-0.27380125414660460D 01
0.35392094279590580D 03
-0.10777033822814850D 05
0.13493737947363220D 06
-0.86387751199106730D 06
0.31600456547536580D 07
-0.70029225301624410D 07
0.95802989549627140D 07
-0.79148578553086700D 07
0.36222321837948840D 07
-0.70543199999999980D 06

(N = 11)

I = 8

-0.41588353129710050D 01
0.53167035015198220D 03
-0.15630274065580460D 05
0.18477591316176540D 06
-0.11048947740846370D 07
0.37975207993617690D 07
-0.79857589728738290D 07
0.10464299332738090D 08
-0.83456623626341160D 07
0.37102535103093070D 07
-0.70543199999999980D 06

I = 9

-0.74115800054566390D 01
0.92339704254301420D 03
-0.24951847165799960D 05
0.26020688357447880D 06
-0.14100020778751850D 07
0.45063438282596450D 07
-0.89781521209175620D 07
0.11293527545564720D 08
-0.87271536476687060D 07
0.37846962947980530D 07
-0.70543199999999980D 06

I = 10

-0.17708925439165380D 02
0.20239721177599240D 04
-0.40128918809659610D 05
0.35295803072169190D 06
-0.17264780148669660D 07
0.51628244414053760D 07
-0.98264624022726930D 07
0.11961628235470690D 08
-0.90209578666288160D 07
0.38400411719398010D 07
-0.70543199999999980D 06

COEFFICIENTS OF THE POLYNOMIALS
$P_n^*(x)/(x-x_i)$ (continued)

(N = 11)

I = 11

-0.9186388296216921OD 02
0.3687059558119203OD 04
-0.5538845018609602OD 05
0.4291467053564157OD 06
-0.1957003365846808OD 07
0.5604857483062362OD 07
-0.1036725413571096OD 08
0.1237092919604665OD 08
-0.9195648538788834OD 07
0.3872196899386644OD 07
-0.7054319999999998OD 06

N = 12

I = 1

-0.1009305478508142OD 01
0.1564329567744069OD 03
-0.5904000051403197OD 04
0.9507253860481013OD 05
-0.8133260698343667OD 06
0.4125606715758665OD 07
-0.1314875665078365OD 08
0.2700754030611674OD 08
-0.3567653604233050OD 08
0.2925781170323776OD 08
-0.1354571146076895OD 08
0.2704156000000000OD 07

I = 2

-0.1050355482063672OD 01
0.1627522086673305OD 03
-0.6137487354608353OD 04
0.9869404032869054OD 05
-0.8426014280238509OD 06
0.4262651954230854OD 07
-0.1353959059258807OD 08
0.2769546394182949OD 08
-0.3640499128109136OD 08
0.2968263495786647OD 08
-0.1365042094824111OD 08
0.2704156000000000OD 07

(N = 12)

I = 3

-0.1130005638607494OD 01
0.1750039667372019OD 03
-0.6589058390927835OD 04
0.1056678912038371OD 06
-0.8986167662107958OD 06
0.4522598097786503OD 07
-0.1427257903654298OD 08
0.2896737468926243OD 08
-0.3772837433101939OD 08
0.3043808686400502OD 08
-0.1383188953208070OD 08
0.2704156000000000OD 07

I = 4

-0.1259987008590981OD 01
0.1949704059902140OD 03
-0.7321821791679306OD 04
0.1168992991703639OD 06
-0.9878306973248931OD 06
0.4930411442909359OD 07
-0.1540047414455643OD 08
0.3087827714574998OD 08
-0.3966045084949159OD 08
0.3150490307626599OD 08
-0.1407875831500404OD 08
0.2704156000000000OD 07

I = 5

-0.1462168404182551OD 01
0.2259603346011598OD 03
-0.8451391373326651OD 04
0.1340056998158005OD 06
-0.1121328615031722OD 07
0.5526364011556226OD 07
-0.1700029864345043OD 08
0.3349388776808684OD 08
-0.4220002573862569OD 08
0.3284699069706707OD 08
-0.1437552112249751OD 08
0.2704156000000000OD 07

COEFFICIENTS OF THE POLYNOMIALS $P_n^*(x)/(x-x_i)$ (continued)

(N = 12)

I = 6

-0.17774090110084440D 01
0.27411662295560000D 03
-0.10187901165549110D 05
0.15981057468493100D 06
-0.13172190226655770D 07
0.63696597527710040D 07
-0.19166647853966690D 08
0.36864614913043360D 08
-0.45307106764057200D 08
0.34406182911952610D 08
-0.14703532663486610D 08
0.27041560000000000D 07

I = 7

-0.22863241685604960D 01
0.35143929209705040D 03
-0.12928158809266100D 05
0.19930308733532540D 06
-0.16040779780371220D 07
0.75375947230744230D 07
-0.21984244415561060D 08
0.40977906012483700D 08
-0.48875309586617870D 08
0.36099436277086250D 08
-0.15042183336513350D 08
0.27041560000000000D 07

I = 8

-0.31637134669490470D 01
0.48353021794298810D 03
-0.17471512020309010D 05
0.26141286017492700D 06
-0.20231540762048540D 07
0.91043508787125080D 07
-0.25464049880790580D 08
0.45694292253791910D 08
-0.52710180022588890D 08
0.37820359472091660D 08
-0.15370194877502440D 08
0.27041560000000000D 07

(N = 12)

I = 9

-0.48463460447902640D 01
0.73254291300179080D 03
-0.25556997895957640D 05
0.36126118341389720D 06
-0.26152764443786540D 07
0.11076903293286250D 08
-0.29447526345421890D 08
0.50691663294906840D 08
-0.56525292661719900D 08
0.39445899926225400D 08
-0.15666957684995930D 08
0.27041560000000000D 07

I = 10

-0.86919741157220160D 01
0.12803975480242770D 04
-0.41074814193766260D 05
0.51304538720347610D 06
-0.33712222550908350D 07
0.13295884596250260D 08
-0.33527129359225730D 08
0.55456243295603750D 08
-0.59965118501394010D 08
0.40847771543197920D 08
-0.15913826467919280D 08
0.27041560000000000D 07

I = 11

-0.20858810712881270D 02
0.28188844868536790D 04
-0.66479439208806720D 05
0.70128491360441830D 06
-0.41637333025608900D 07
0.15376337170479350D 08
-0.37061910459839990D 08
0.59351358387366690D 08
-0.62655520668857870D 08
0.41907005475455270D 08
-0.16095295051758860D 08
0.27041560000000000D 07

COEFFICIENTS OF THE POLYNOMIALS

$P_n^*(x)/(x-x_i)$ (continued)

(N = 12)

I = 12

-0.10846360047086500D 03
0.51559690463514850D 04
-0.92197417744400200D 05
0.85714252445951830D 06
-0.47460933446364170D 07
0.16790453363184500D 08
-C.39337864617272180D 08
0.61762577991757680D 08
-0.64272493552206250D 08
0.42529277095548140D 08
-0.16200004539231030D 08
0.27041560000000000D 07

N = 13

I = 1

-0.10079715286689510D 01
0.18243480916835120D 03
-0.80713976160663870D 04
0.15330097897999020D 06
-0.15607421072745860D 07
0.95417344384769900D 07
-0.37311857364665380D 08
0.96475435981292820D 08
-0.16673481171541170D 09
0.19042647081912740D 09
-0.13786672036695810D 09
0.57285552860546860D 08
-0.10400599999999980D 08

(N = 13)

I = 2

-0.10429712616983220D 01
0.18873297851273380D 03
-0.83450914693393850D 04
0.15833858496354280D 06
-0.16096815846966000D 07
0.98220090495688290D 07
-0.38315116413652590D 08
0.98778976478833180D 08
-0.17012180818179740D 09
0.19350609793310380D 09
-0.13944189578075440D 09
0.57631813044520720D 08
-0.10400599999999980D 08

I = 3

-0.11101378477288740D 01
0.20081267981965990D 03
-0.88690991095794720D 04
0.16795375306639010D 06
-0.17026697369630140D 07
0.10351309566871580D 08
-0.40194985295797750D 08
0.10305335658962990D 09
-0.17633237685047210D 09
0.19907316154864720D 09
-0.14224164371270630D 09
0.58235153454759510D 08
-0.10400599999999980D 08

I = 4

-0.12177677100408040D 01
0.22015076567274520D 03
-0.97054250797130950D 04
0.18321872369925510D 06
-0.18491574719641470D 07
0.11176499795303420D 08
-0.43087072104590820D 08
0.10952267673582370D 09
-0.18554999183803190D 09
0.20714875218926850D 09
-0.14619803585958360D 09
0.59063190730108370D 08
-0.10400599999999980D 08

COEFFICIENTS OF THE POLYNOMIALS

$P_n^*(x)/(x-x_i)$ (continued)

(N = 13)

I = 5

```
-0.13807456076844240D 01
 0.24938924169703470D 03
-0.10963963406266510D 05
 0.20600177182381960D 06
-0.20651787550969150D 07
 0.12374017389442320D 08
-0.47200090695790450D 08
 0.11850142742212430D 09
-0.19798539202253470D 09
 0.21770203249501490D 09
-0.15119280842328430D 09
 0.60071303146785120D 08
-0.10400599999999980D 08
```

I = 6

```
-0.16254106083538880D 01
 0.29318277109067540D 03
-0.12835570496802190D 05
 0.23946269059726430D 06
-0.23767360347638980D 07
 0.14060256821493990D 08
-0.52823008723021120D 08
 0.13036006216061530D 09
-0.21379280571123440D 09
 0.23058480315709960D 09
-0.15704404056613840D 09
 0.61205147619538480D 08
-0.10400599999999980D 08
```

I = 7

```
-0.20000000000000000D 01
 0.35999999999999990D 03
-0.15659999999999980D 05
 0.28899999999999990D 06
-0.28253999999999980D 07
 0.16403231999999980D 08
-0.60310559999999990D 08
 0.14542751999999990D 09
-0.23292821999999980D 09
 0.24545415999999970D 09
-0.16349743199999980D 09
 0.62403599999999980D 08
-0.10400599999999980D 08
```

(N = 13)

I = 8

```
-0.25989495325629240D 01
 0.46625427624523020D 03
-0.20073625338025120D 05
 0.36407741793960490D 06
-0.34764135843287480D 07
 0.19623634618423160D 08
-0.70002386979225560D 08
 0.16379082336259800D 09
-0.25495904554972060D 09
 0.26170448342966820D 09
-0.17022701675121470D 09
 0.63602052380461440D 08
-0.10400599999999980D 08
```

I = 9

```
-0.36264255887223260D 01
 0.64685849459456770D 03
-0.27354641374456070D 05
 0.48160875083736360D 06
-0.44245701265234570D 07
 0.23943278663642210D 08
-0.82012460865595080D 08
 0.18499071129263050D 09
-0.27887545939564370D 09
 0.27843637866464460D 09
-0.17684807380864750D 09
 0.64735896853214790D 08
-0.10400599999999980D 08
```

I = 10

```
-0.55920489476264360D 01
 0.98648189703509230D 03
-0.40282425826881880D 05
 0.67036126249861210D 06
-0.57672967016894390D 07
 0.29412607766919960D 08
-0.95880735722672390D 08
 0.20770874136904660D 09
-0.30299336534800010D 09
 0.29448811625060740D 09
-0.18294253782839130D 09
 0.65744009269891540D 08
-0.10400599999999980D 08
```

COEFFICIENTS OF THE POLYNOMIALS

$P_n^*(x)/(x-x_i)$ (continued)

(N = 13)

I = 11

-0.10079532080864670D 02
0.17328778717481680D 04
-0.65084769641774720D 05
0.95831383449131930D 06
-0.74929847037156350D 07
0.35621381825677550D 08
-0.11024115446467420D 09
0.22964364791696370D 09
-0.32504456994910890D 09
0.30854183556664090D 09
-0.18809455571035100D 09
0.66572046545240380D 08
-0.10400599999999980D 08

I = 12

-0.24271373142518990D 02
0.38282903577150800D 04
-0.10586468226722040D 06
0.13178219165339170D 07
-0.93172482038267870D 07
0.41498412167982510D 08
-0.12281557116129330D 09
0.24778035238250510D 09
-0.34250008457189330D 09
0.31929514242580010D 09
-0.19193155229102610D 09
0.67175386955479200D 08
-0.10400599999999980D 08

I = 13

-0.12644666618088360D 03
0.70245338566609130D 04
-0.14736930837191630D 06
0.16173403145188670D 07
-0.10667000989156780D 08
0.45522697896197000D 08
-0.13097524020902070D 09
0.25909930830793510D 09
-0.35307976886615020D 09
0.32566732552037570D 09
-0.19416523890094200D 09
0.67521647139453030D 08
-0.10400599999999930D 08

N = 14

I = 1

-0.10069054782458360D 01
0.21043628670453990D 03
-0.10783518111968030D 05
0.23837125475372330D 06
-0.28441945032009650D 07
0.20576174872216570D 08
-0.96481786447879590D 08
0.30468070295618890D 09
-0.66011574395699110D 09
0.98263765909997460D 09
-0.98735094411646760D 09
0.63952860422322550D 09
-0.24097472348001660D 09
0.40116599999999990D 08

I = 2

-0.10371104684228440D 01
0.21671759964702230D 03
-0.11100486193258690D 05
0.24519315203176540D 06
-0.29224391985846560D 07
0.21112267750251590D 08
-0.98820046304836740D 08
0.31139543910837490D 09
-0.67295388233064620D 09
0.99880012110107130D 09
-0.10002071037698320D 10
0.64537983781746240D 08
-0.24213507457301090D 09
0.40116599999999990D 08

COEFFICIENTS OF THE POLYNOMIALS

$P_n^*(x)/(x-x_i)$ (continued)

(N = 14)

I = 3

-0.10945701042210040D 01
0.22866164573420590D 03
-0.11702419723639010D 05
0.25811888195511410D 06
-0.30702047934305350D 07
0.22120224016802250D 08
-0.10319175583517800D 09
0.32386273118846210D 09
-0.69659163832994400D 09
0.10282640822576100D 10
-0.10233718266634350D 10
0.65578080356590450D 09
-0.24416564786193610D 09
0.40116599999999990D 08

I = 4

-0.11853306384700660D 01
0.24751442830826710D 03
-0.12650424287634700D 05
0.27839810745672320D 06
-0.33007450983788320D 07
0.23681141365139410D 08
-0.10989809578462980D 09
0.34276783051801440D 09
-0.73194343166031770D 09
0.10716230071987210D 10
-0.10568339927924810D 10
0.67049367136484310D 09
-0.24697197272741550D 09
0.40116599999999990D 08

(N = 14)

I = 5

-0.13199153962437980D 01
0.27544005597036710D 03
-0.14049918526392390D 05
0.30816075444191710D 06
-0.36362339215554540D 07
0.25927127214786090D 08
-0.11941162739159840D 09
0.36912929785643290D 09
-0.78025480169954270D 09
0.11295302404722740D 10
-0.11003963250961380D 10
0.68912423001244720D 09
-0.25042288827733710D 09
0.40116599999999990D 08

I = 6

-0.15161710607407580D 01
0.31609714612803210D 03
-0.16077330637859610D 05
0.35090667757174570D 06
-0.41120883805108430D 07
0.29060704866359880D 08
-0.13241567161503020D 09
0.40429806385895040D 09
-0.84294751415744180D 09
0.12024250314431100D 10
-0.11534911543210250D 10
0.71108793568535140D 09
-0.25435704837033370D 09
0.40116599999999990D 08

COEFFICIENTS OF THE POLYNOMIALS
$P_n^*(x)/(x-x_i)$ (continued)

(N = 14)

I = 7

-0.18049646403537720D 01
0.37578467674825420D 08
-0.19031935799849300D 05
0.41241287621467630D 06
-0.47843243277376570D 07
0.33382697661047380D 08
-0.14983658064213530D 09
0.44986285138408280D 09
-0.92126560909521140D 09
0.12900962569385440D 10
-0.12149596007792130D 10
0.73558445760512930D 09
-0.25859050142234330D 09
0.40116599999999990D 08

I = 8

-0.22422905954241500D 01
0.46585315789012280D 03
-0.23441235145503930D 05
0.50244970703015500D 06
-0.57416323780458560D 07
0.39324448510673690D 08
-0.17281744290778780D 09
0.50733061810099180D 09
-0.10156201690733830D 10
0.13911041045129520D 10
-0.12828439390025380D 10
0.76159324053700740D 09
-0.26292529857765640D 09
0.40116599999999990D 08

(N = 14)

I = 9

-0.29373422408393690D 01
0.60821389113244550D 03
-0.30289244915877940D 05
0.63808107291382380D 06
-0.71229930356286800D 07
0.47456473659044190D 08
-0.20249980523924240D 09
0.57740263089002750D 09
-0.11246104691522120D 10
0.15021683094537710D 10
-0.13542585457758150D 10
0.78789815524134710D 09
-0.26715875162966600D 09
0.40116599999999990D 08

I = 10

-0.41258264545699580D 01
0.84940111152544690D 03
-0.41549543307381450D 05
0.84979835908124590D 06
-0.91315334289765970D 07
0.58371048055375070D 08
-0.23940203698940500D 09
0.65877451818192800D 09
-0.12439151879965760D 10
0.16177332191812960D 10
-0.14254071787365220D 10
0.81314437068440010D 09
-0.27109291172266260D 09
0.40116599999999990D 08

COEFFICIENTS OF THE POLYNOMIALS

$P_n^*(x)/(x-x_i)$ (continued)

(N = 14)

I = 11

-0.63957614994190500D 01
0.13022041497201630D 04
-0.61513128408473890D 05
0.11896555879521140D 07
-0.11981847811414090D 08
0.72255528335757830D 08
-0.28231372004949410D 09
0.74676425157665550D 09
-0.13655274325948350D 10
0.17299700043088120D 10
-0.14917937922113510D 10
0.83592479863585380D 09
-0.27454382727258420D 09
0.40116599999999990D 08

I = 12

-0.11574162157586150D 02
0.22966128234427870D 04
-0.99808481529113370D 05
0.17096370673251690D 07
-0.15664736490572830D 08
0.88131683554205400D 08
-0.32713902258233050D 09
0.83257505630568550D 09
-0.14779539988498060D 10
0.18294164821220520D 10
-0.15486318536737270D 10
0.85488782922267090D 09
-0.27735015213806370D 09
0.40116599999999990D 08

(N = 14)

I = 13

-0.27946576521975180D 02
0.50877699303161640D 04
-0.16299086393618340D 06
0.23622999693438930D 07
-0.19583843820272490D 08
0.10327411926099290D 09
-0.36671946922631620D 09
0.90416930565976880D 09
-0.15678030641774040D 10
0.19062727573007680D 10
-0.15914037904035320D 10
0.86885374294133100D 09
-0.27938072542698890D 09
0.40116599999999990D 08

I = 14

-0.14581307272487430D 03
0.93592930947540420D 04
-0.22757146947683940D 06
0.29087565319279450D 07
-0.22499462811690370D 08
0.11370404087734690D 09
-0.39256365898413480D 09
0.94914118241443470D 09
-0.16226264769256810D 10
0.19521312846090240D 10
-0.16164795526579080D 10
0.87592672246302460D 09
-0.28054107651998310D 09
0.40116599999999990D 08

COEFFICIENTS OF THE POLYNOMIALS

$P_n^*(x)/(x-x_i)$ (continued)

N = 15

I = 1

-0.1006040152477297OD 01
0.2404374842270507OD 03
-0.1412436152400878OD 05
0.3593128598206432OD 06
-0.4961212977893652OD 07
0.4184854685824095OD 08
-0.2311304203202154OD 09
0.8715527969314596OD 09
-0.2297410852775661OD 10
0.4272296428411988OD 10
-0.5577274362288000OD 10
0.4998946417222674OD 10
-0.2928290580968200OD 10
0.1009195165413053OD 10
-0.1551175199999999OD 09

I = 2

-0.1032378180455984OD 01
0.2467051091053398OD 03
-0.1448776283525810D 05
0.3683448346692362OD 06
-0.5081776464426468OD 07
0.4281970288579122OD 08
-0.2361789698982142OD 09
0.8891586220925500OD 09
-0.2339382739469069OD 10
0.4340817147465610OD 10
-0.5652551832913297OD 10
0.5052107344562644OD 10
-0.2950072990881503OD 10
0.1013128877904406OD 10
-0.1551175199999999OD 09

(N = 15)

I = 3

-0.1082129560452869OD 01
0.2585402222717774OD 03
-0.1517304386163902OD 05
0.3853540608877714OD 06
-0.5308265843378288OD 07
0.4463813355810491OD 08
-0.2455928774186127OD 09
0.9218209373645729OD 09
-0.2416774913239741OD 10
0.4466256795973893OD 10
-0.5789211587109770OD 10
0.5147689501367660OD 10
-0.2988807445470494OD 10
0.1020036789166849OD 10
-0.1551175199999999OD 09

I = 4

-0.1159811678133904OD 01
0.2770096641646986OD 03
-0.1624083315913267OD 05
0.4117786083779396OD 06
-0.5658676900490553OD 07
0.4743611058470021OD 08
-0.2599778526268026OD 09
0.9713108049667565OD 09
-0.2532866306484425OD 10
0.4652226824821226OD 10
-0.5989105030955195OD 10
0.5285382174132820OD 10
-0.3043664047454706OD 10
0.1029637699033690OD 10
-0.1551175199999999OD 09

COEFFICIENTS OF THE POLYNOMIALS
$P_n^*(x)/(x-x_i)$ (continued)

(N = 15)

I = 5

-0.12730970773184230D 01
0.30392251000383360D 03
-0.17792902679860220D 05
0.45002337213037700D 06
-0.61627019399481760D 07
0.51427785627769750D 08
-0.28028954072007160D 09
0.10403253648001450D 10
-0.26924031654720320D 10
0.49035297017800650D 10
-0.62541611670470500D 10
0.54641925653287200D 10
-0.31133126064361470D 10
0.10415387462979740D 10
-0.15511751999999990D 09

I = 6

-0.14345645273585880D 01
0.34223749949777700D 03
-0.19994618991286290D 05
0.50394152936850410D 06
-0.68669710511210130D 07
0.56940074737338770D 08
-0.30793105169337000D 09
0.11326167563681360D 10
-0.29014838853350630D 10
0.52255067461657440D 10
-0.65854823344066340D 10
0.56819159222852080D 10
-0.31958357783530470D 10
0.10552527502683340D 10
-0.15511751999999990D 09

(N = 15)

I = 7

-0.16650098603607240D 01
0.39683010652589540D 03
-0.23115614641881100D 05
0.57969713154261050D 06
-0.78439327865484000D 07
0.64460155784839010D 08
-0.34487542895894950D 09
0.12530451403345920D 10
-0.31670576123719550D 10
0.56227381145203220D 10
-0.69819660060510230D 10
0.59344986988532170D 10
-0.32886568640604680D 10
0.10702182755502340D 10
-0.15511751999999990D 09

I = 8

-0.20000000000000000D 01
0.47600000000000000D 03
-0.27607999999999990D 05
0.68734400000000000D 06
-0.92067919999999990D 07
0.74703439999999970D 08
-0.39377575999999980D 09
0.14073497599999980D 10
-0.34956416599999970D 10
0.60968317199999950D 10
-0.74385091199999960D 10
0.62153985599999960D 10
-0.33885154799999980D 10
0.10858226399999980D 10
-0.15511751999999990D 09

COEFFICIENTS OF THE POLYNOMIALS

$P_n^*(x)/(x-x_i)$ (continued)

(N = 15)

I = 9

-0.25037371216725420D 01
0.59462820966565230D 03
-0.34264573377579380D 05
0.84379803426575600D 06
-0.11133973678256590D 08
0.88693731620030380D 08
-0.45792748152110470D 09
0.16011978721570450D 10
-0.38907391799828800D 10
0.66432116239475960D 10
-0.79440440592891930D 10
0.65151354569540330D 10
-0.34915136019074060D 10
0.11014270044497620D 10
-0.15511751999999990D 09

I = 10

-0.33011544884820390D 01
0.78137945630398640D 03
-0.44561031797570630D 05
0.10785497883883430D 07
-0.13905090623848970D 08
0.10779398854364180D 09
-0.54072029595592120D 09
0.18378528741743480D 10
-0.43486692904945690D 10
0.72473157080254900D 10
-0.84799085038268430D 10
0.68211751770212770D 10
-0.35932443448646990D 10
0.11163925297316650D 10
-0.15511751999999990D 09

(N = 15)

I = 11

-0.40017022773344250D 01
0.10970770784429030D 04
-0.61454861805635260D 05
0.14443125523029000D 07
-0.17930899592615290D 08
0.13345340595477830D 09
-0.64395782849385780D 09
0.21140484721628930D 10
-0.48534013477272510D 10
0.78813357285103280D 10
-0.90192310519957370D 10
0.71183137806629320D 10
-0.36890032245624610D 10
0.11301065337020220D 10
-0.15511751999999990D 09

I = 12

-0.72573609683551000D 01
0.16890973441823730D 04
-0.91376725490805910D 05
0.20313590993403970D 07
-0.23654503729479240D 08
0.16622265564157130D 09
-0.76469843200354300D 09
0.24149028864505400D 10
-0.53723899378805530D 10
0.85032145783714720D 10
-0.95279838438689320D 10
0.73896231506956810D 10
-0.37740682800167090D 10
0.11420075809663060D 10
-0.15511751999999990D 09

COEFFICIENTS OF THE POLYNOMIALS

$P_n^*(x)/(x-x_i)$ (continued)

(N = 15)

I = 13

-0.1317580198756245OD 02
0.2988590718999853OD 04
-0.1487733728469989OD 06
0.2931703260288643OD 07
-0.3108220096382503OD 08
0.2039128097105898OD 09
-0.8917186503865704OD 09
0.2710683957654092OD 10
-0.5856467853908762OD 10
0.9059545289113264OD 10
-0.9968034291398928OD 10
0.7617907569263935OD 10
-0.3844023506301435OD 10
0.1151608490833147OD 10
-0.1551175199999999OD 09

I = 14

-0.3188439605687363OD 02
0.6635640341738092OD 04
-0.2437357909452096OD 06
0.4066670076263621OD 07
-0.3902873024971776OD 08
0.2400825435478175OD 09
-0.1004638307668875OD 10
0.2959265163007749OD 10
-0.6246326223911842OD 10
0.9492982563001040OD 10
-0.1030196688024276OD 11
0.7787493349471583OD 10
-0.3895091898124201OD 10
0.1158516402095590OD 10
-0.1551175199999999OD 09

(N = 15)

I = 15

-0.1665628150358329OD 08
0.1223190425593861OD 05
-0.3411365925937650OD 06
0.5020770792354365OD 07
-0.4496683119844917OD 08
0.2651148349447862OD 09
-0.1078741582333888OD 10
0.3116233311535108OD 10
-0.6485502710946163OD 10
0.9753053829891920OD 10
-0.1049886232860655OD 11
0.7886094932177570OD 10
-0.3924447750598505OD 10
0.1162450114586946OD 10
-0.1551175199999999OD 09

APPENDIX G

ELEMENTS OF THE EXPLICIT INVERSE MATRIX WITH DIVISION BY WEIGHTS

From equations (37) and (45) it is evident that the explicit inverse matrix with division by weights is the necessary matrix for use in obtaining the temperature function:

$$\alpha(x_i) = u(-\log x_i) = \sum_{\kappa=0}^{N-1} F(\kappa+1) \left[\frac{q_{\kappa j}}{A_i^*}\right] .$$

ELEMENTS OF THE EXPLICIT INVERSE
MATRIX WITH DIVISION BY WEIGHTS

N = 3

I = 1

0.67620999227555400D 00
-0.73524199845511530D 01
0.12000000000000080D 02

I = 2

-0.14999999999999990D 01
0.14999999999999990D 02
-0.14999999999999990D 02

I = 3

0.53237900077244490D 01
-0.16647580015448840D 02
0.11999999999999990D 02

N = 4

I = 1

-0.65496972587504130D 00
0.12395559842403600D 02
-0.45626858219032430D 02
0.42664577893729350D 02

I = 2

0.12290561919335860D 01
-0.22746685660881600D 02
0.76664299881176180D 02
-0.57641919954206000D 02

I = 3

-0.24952495419778680D 01
0.42343845761147220D 02
-0.96261459981441690D 02
0.57641919954205890D 02

I = 4

0.87783059330621940D 01
-0.49135573227363170D 02
0.82366875462154930D 02
-0.42664577893729010D 02

N = 5

I = 1

0.64457574561626680D 00
-0.18660971287045070D 02
0.11578146204084730D 03
-0.23948231451334420D 03
0.15481333922979940D 03

I = 2

-0.11196222405308090D 01
0.32133165655799910D 02
-0.19334776617762280D 03
0.37563762838230080D 03
-0.21703556145201720D 03

I = 3

0.18749999999999980D 01
-0.52499999999999990D 02
0.28874999999999980D 03
-0.47249999999999990D 03
0.23625000000000010D 03

I = 4

-0.37321558320700430D 01
0.95791727360611900D 02
-0.36864824974282230D 03
0.49250461742576680D 03
-0.21703556145201670D 03

I = 5

0.13096091215873580D 02
-0.11370836617381250D 03
0.32621455387960370D 03
-0.37977104240584490D 03
0.15481333922979590D 03

ELEMENTS OF THE EXPLICIT INVERSE
MATRIX WITH DIVISION BY WEIGHTS

N = 6

I = 1

-0.638702891448609000D 00
0.261644989649310500D 02
-0.241176392442383800D 03
0.823416513896431500D 03
-0.115972315980272600D 04
0.570234526257290800D 03

I = 2

0.106330622975233000D 01
-0.433787024614608700D 02
0.394363168199260100D 03
-0.131156402546257500D 04
0.177036741131210600D 04
-0.816064921773318000D 03

I = 3

-0.162092929530268200D 01
0.654617137754321300D 02
-0.575089190220821300D 03
0.179456395767370700D 04
-0.220824253926463500D 04
0.927563925494165200D 03

I = 4

0.263693816254427300D 01
-0.103824676767192000D 03
0.834786702154134200D 03
-0.223723305555681100D 04
0.242957708820618700D 04
-0.927563925494164300D 03

I = 5

-0.521376395623408500D 01
0.188199406068825000D 03
-0.107877365804899000D 04
0.239074359794730200D 04
-0.230995719755446000D 04
0.816064921773309800D 03

(N = 6)

I = 6

0.182772839820937600D 02
-0.226341247844999300D 03
0.973070453003434300D 03
-0.188686913725838700D 04
0.169144147148368600D 04
-0.570234526257277000D 03

N = 7

I = 1

0.635056718620002000D 00
-0.349115379064297600D 02
0.444279785352146900D 03
-0.221135808705366300D 04
0.506580745487558100D 04
-0.536418536923828600D 04
0.212405463278323900D 04

I = 2

-0.103015607132200600D 01
0.565056936267359900D 02
-0.713906021180589900D 03
0.350679533616941500D 04
-0.787104733811230000D 04
0.809432937424546800D 04
-0.307858795399858800D 04

I = 3

0.146785642402655400D 01
-0.812032879021264100D 02
0.100929707798274000D 04
-0.481313887521106500D 04
0.103374173382135400D 05
-0.100396895432109500D 05
0.358934988926270100D 04

ELEMENTS OF THE EXPLICIT INVERSE MATRIX WITH DIVISION BY WEIGHTS

(N = 7)

I = 4

-0.21874999999999980D 01
0.11812500000000000D 03
-0.14175000000000000D 04
0.63524999999999990D 04
-0.12560624999999990D 05
0.11261249999999980D 05
-0.37537500000000020D 04

I = 5

0.35204555887100010D 01
-0.18529521480577540D 03
0.20377373884617530D 04
-0.79266328307875080D 04
0.13979217961099200D 05
-0.11496409792365190D 05
0.35893498892626960D 04

I = 6

-0.69410653211762830D 01
0.33499054543956470D 03
-0.26553296088702320D 04
0.86058593537967680D 04
-0.13578219776863660D 05
0.10377198349745970D 05
-0.30785879539985680D 04

I = 7

0.24321935531217310D 02
-0.40620454164723460D 03
0.24240160528103890D 04
-0.68911106957305150D 04
0.10105700100432590D 05
-0.73801424274610320D 04
0.21240546327832080D 04

N = 8

I = 1

-0.63263629050388220D 00
0.44904361134180810D 02
-0.75130772461264850D 03
0.50790321475309150D 04
-0.16738927934245450D 05
0.28517349619132120D 05
-0.24099556251708940D 05
0.79803684875954930D 04

I = 2

0.10088738950655580D 01
-0.71515869608273520D 02
0.11915715970486080D 04
-0.79955698673191210D 04
0.26057030694553270D 05
-0.43705585221165210D 05
0.36178290078771380D 05
-0.11664144752831990D 05

I = 3

-0.14081395444254550D 01
0.99539951323618810D 02
-0.16437571873641720D 04
0.10856214613981250D 05
-0.34559345638578920D 05
0.56179519216495980D 05
-0.44749661063421930D 05
0.13823425769235650D 05

I = 4

0.19506271541102560D 01
-0.13714860269817760D 03
0.22260096024338430D 04
-0.14261847226419680D 05
0.43486763859943770D 05
-0.67093136369321480D 05
0.50629390880600230D 05
-0.14854809783306000D 05

ELEMENTS OF THE EXPLICIT INVERSE

MATRIX WITH DIVISION BY WEIGHTS

(N = 8)

I = 5

```
-0.28270116134008710D 01
 0.19662068346251010D 03
-0.30804548507999980D 04
 0.18576680283564960D 05
-0.52456397193370430D 05
 0.75267796535146170D 05
-0.53354277602541830D 05
 0.14854809783305980D 05
```

I = 6

```
 0.45275221271127710D 01
-0.30689695082148760D 03
 0.44110301906861850D 04
-0.23240704879435370D 05
 0.58913236415820400D 05
-0.77973493989913020D 05
 0.52014319321227460D 05
-0.13823425769235560D 05
```

I = 7

```
-0.89144666562789340D 01
 0.55415840227710460D 03
-0.57814946771425290D 04
 0.25502564074450390D 05
-0.58041610578821550D 05
 0.71582884558008030D 05
-0.45470723191052030D 05
 0.11664144752831820D 05
```

I = 8

```
 0.31230068535155660D 02
-0.67566361609512550D 03
 0.53201117676994600D 04
-0.20618588633532940D 05
 0.43667373451622040D 05
-0.51507750348382530D 05
 0.31763023161458740D 05
-0.79803684875953020D 04
```

N = 9

I = 1

```
 0.63094690159499350D 00
-0.56144067147718930D 02
 0.11922225317072080D 04
-0.10448389121829060D C5
 0.46224589234221030D 05
-0.11218523386107140D 06
 0.15126259454114080D 06
-0.10613953845652780D 06
 0.30188269946895430D 05
```

I = 2

```
-0.99434922630698750D 00
 0.85408279628242270D 02
-0.18725077842749480D 04
 0.16335839433292440D 05
-0.71786191232140630D 05
 0.17262944053493840D 06
-0.22999798830007930D 06
 0.15897455931633070D 06
-0.44381700059841140D 05
```

I = 3

```
 0.13561217085072470D 01
-0.12036985085193770D 03
 0.25359056832479280D 04
-0.21917518679328140D 05
 0.95003168642251510D 05
-0.22431467127417700D 06
 0.29207121661956960D 06
-0.19644195852830700D 06
 0.53188530258027430D 05
```

I = 4

```
-0.18076013195316310D 01
 0.15995412571003360D 03
-0.33374743057706110D 04
 0.28363935208562880D 05
-0.12000917873689330D 06
 0.27472299293684350D 06
-0.34504174828970990D 06
 0.22333117187123050D 06
-0.58191387545852760D 05
```

ELEMENTS OF THE EXPLICIT INVERSE
MATRIX WITH DIVISION BY WEIGHTS

(N = 9)

I = 5

```
 0.24609374999999980D 01
-0.21656250000000000D 03
 0.44395312499999990D 04
-0.36599062500000000D 05
 0.14850773437499990D 06
-0.32376093749999990D 06
 0.38710546874999970D 06
-0.23930156249999990D 06
 0.59825390625000010D 05
```

I = 6

```
-0.35423371992380150D 01
 0.30832612996597780D 03
-0.61012780666898300D 04
 0.47404503239454910D 05
-0.17882655111494290D 06
 0.36429059007332490D 06
-0.41108239647497120D 06
 0.24219992849559070D 06
-0.58191387545852670D 05
```

I = 7

```
 0.56589921408547840D 01
-0.48003575926795000D 03
 0.87216161808651660D 04
-0.59461921498086270D 05
 0.20222663113608010D 06
-0.38138919379832560D 06
 0.40625635414618650D 06
-0.22906628353591180D 06
 0.53188530258027340D 05
```

I = 8

```
-0.11134161372277090D 02
 0.86626630780323180D 03
-0.11479412061649910D 05
 0.65739913426994570D 05
-0.20121824117593400D 06
 0.35426794697368490D 06
-0.35986367476130570D 06
 0.19607904116239290D 06
-0.44381700059839920D 05
```

(N = 9)

I = 9

```
 0.39001694289452010D 02
-0.10602788886162110D 04
 0.10622421003920810D 05
-0.53508791074806880D 05
 0.15253238835015280D 06
-0.25700314282479870D 06
 0.25355738385848900D 06
-0.13536662111862080D 06
 0.30188269946892230D 05
```

N = 10

I = 1

```
-0.62972078889511680D 00
 0.68631241594361980D 02
-0.18007322514163060D 04
 0.19787480981637290D 05
-0.11232455134290630D 06
 0.36273559299210630D 06
-0.69145818794950720D 06
 0.76901847430852180D 06
-0.46080521705457350D 06
 0.11482677561427350D 06
```

I = 2

```
 0.98397308015031300D 00
-0.10718187566752850D 03
 0.28074636043399860D 04
-0.30759373574475980D 05
 0.17385617851724700D 06
-0.55819292467264940D 06
 0.10561498767431730D 07
-0.11637950926080640D 07
 0.68955600212862640D 06
-0.16952953248708180D 06
```

ELEMENTS OF THE EXPLICIT INVERSE
MATRIX WITH DIVISION BY WEIGHTS

(N = 10)

I = 3

-0.13200922820004000D 01
0.14363806009058600D 03
-0.37496162693665910D 04
0.40840168876119880D 05
-0.22886025683627430D 06
0.72643928184494650D 06
-0.13548600784180940D 07
0.14672828405019330D 07
-0.85202751523799080D 06
0.20479977286030410D 06

I = 4

0.17134865755651150D 01
-0.18609471533044610D 03
0.48294021180310210D 04
-0.52068450405179270D 05
0.28754150696226020D 06
-0.89548791099262010D 06
0.16320721886540260D 07
-0.17216549991630270D 07
0.97183826091386580D 06
-0.22688995363410560D 06

I = 5

-0.22391285740396210D 01
0.24240619103871870D 03
-0.62282228225405700D 04
0.66004587961004810D 05
-0.35578415857698580D 06
0.10751126718341510D 07
-0.18939045898428100D 07
0.19286143921177980D 07
-0.10516921388897390D 07
0.23764031359797480D 06

(N = 10)

I = 6

0.30224413129666500D 01
-0.32536632332934910D 03
0.82120953859297140D 04
-0.84433163665506040D 05
0.43694386613728230D 06
-0.12605[illegible]71027702050D 07
0.21208026082988250D 07
-0.26701265705269370D 07
0.10870706834920260D 07
-0.23764031359797450D 06

I = 7

-0.43347755026589230D 01
0.46152442191895890D 03
-0.11245195102879330D 05
0.10907622419615850D 06
-0.52619608282311960D 06
0.14230013082128740D 07
-0.22667976051638040D 07
0.21149872426799040D 07
-0.10701713217930830D 07
0.22688995363410560D 06

I = 8

0.69152893859037290D 01
-0.71754087313121890D 03
0.15062038367213580D 05
-0.13712987612435800D 06
0.59617970742371490D 06
-0.15014489889478440D 07
0.22625302986972490D 07
-0.20238545415689480D 07
0.99117644050474260D 06
-0.20479977286030280D 06

ELEMENTS OF THE EXPLICIT INVERSE
MATRIX WITH DIVISION BY WEIGHTS

(N = 10)

I = 9

-0.1360025147207961OD 02
0.1294447836815844OD 04
-0.2120673629728642OD 05
0.1524392223626234OD 06
-0.5994894793491796OD 06
0.1406575583152212OD 07
-0.2023328440826579OD 07
0.1750410245113953OD 07
-0.8362097902550876OD 06
0.1695295324870771OD 06

I = 10

0.4763681894046310OD 02
-0.1588805827061970OD 04
0.1970331837925889OD 05
-0.1246850438820475OD 06
0.4569357287515543OD 06
-0.1026455998116240OD 07
0.1434574206280863OD 07
-0.1216340659985600OD 07
0.5726357634738024OD 06
-0.1148267756142558OD 06

N = 11

I = 1

0.6288026023091337OD 00
-0.8236622063966381OD 02
0.2614290464291346OD 04
-0.3512282233083765OD 05
0.2477347666349885OD 06
-0.1018472507519457OD 07
0.2565630906080904OD 07
-0.4009767738365648OD 07
0.3787918989673284OD 07
-0.1979145850714310OD 07
0.4387488389408913OD 06

(N = 11)

I = 2

-0.9762922887292349OD 00
0.1278358604482423OD 03
-0.4052807305674417OD 04
0.5434075412351191OD 05
-0.3821779081675776OD 06
0.1565122871153368OD 07
-0.3923365394328944OD 07
0.6094771554113454OD 07
-0.5715837437578226OD 07
0.2960872582056142OD 07
-0.6496173864292751OD 06

I = 3

0.1294009449270308OD 01
-0.1693134140417345OD 03
0.5355579682999507OD 04
-0.7152732987750088OD 05
0.5002032665388329OD 06
-0.2033112905214089OD 07
0.5048560312393995OD 07
-0.7753610579121175OD 07
0.7174706531689347OD 07
-0.3660070795897852OD 07
0.7896722358551489OD 06

I = 4

-0.1647756520258203OD 01
0.2153344699620401OD 03
-0.6785372057335959OD 04
0.9003082319817133OD 05
-0.6236998297670237OD 06
0.2504053055352844OD 07
-0.6124630148778836OD 07
0.9241088289318982OD 07
-0.8382659432613852OD 07
0.4185267337889655OD 07
-0.8828836142475331OD 06

ELEMENTS OF THE EXPLICIT INVERSE

MATRIX WITH DIVISION BY WEIGHTS

(N = 11)

I = 5

0.20925320059004940D 01
-0.27291771296259300D 03
0.85470159711018340D 04
-0.11221276108008450D 06
0.76580444244534230D 06
-0.30163378454905110D 07
0.72126476877351980D 07
-0.10613016080694200D 08
0.93766039635306540D 07
-0.45587730668363300D 07
0.93701110644666970D 06

I = 6

-0.27070312500000030D 01
0.35191406249999990D 03
-0.10909335937500010D 05
0.14076562500000000D 06
-0.93785097656250140D 06
0.35871304218750020D 07
-0.83037642187500140D 07
0.11821497187500010D 08
-0.10116473554687500D 08
0.47740661718750060D 07
-0.95481323437500150D 06

I = 7

0.36368468696400980D 01
-0.47010605445151210D 03
0.14314916797832990D 05
-0.17923460126707820D 06
0.11474710862920090D 07
-0.41974249472418480D 07
0.93018408412828020D 07
-0.12725317995043420D 08
0.10513146152103770D 08
-0.48113379976303560D 07
0.93701110644666820D 06

(N = 11)

I = 8

-0.52049914834825970D 01
0.66541217343476270D 03
-0.19562073819567300D 05
0.23125634510221690D 06
-0.13828313593756070D 07
0.47527882042786310D 07
-0.99945788459844230D 07
0.13096596717268040D 08
-0.10445016032745860D 08
0.40435668045856490D 07
-0.88288361424752910D 06

I - 9

0.82966451112626190D 01
-0.10336659056676870D 04
0.27931509968548430D 05
-0.29127988457733320D 06
0.15783796218432000D 07
-0.50444757345765650D 07
0.10050291820008940D 08
-0.12642161324688600D 08
0.97693199820902740D 07
-0.42366515626535970D 07
0.78967223585514200D 06

I = 10

-0.16312795060826480D 02
0.18644068765925460D 04
-0.36965246445101230D 05
0.32513164279871510D 06
-0.15903665149134350D 07
0.47557994039214340D 07
-0.90517670251564250D 07
0.11018601364003760D 08
-0.83097665883900020D 07
0.35373012822364780D 07
-0.64981738642925160D 06

ELEMENTS OF THE EXPLICIT INVERSE
MATRIX WITH DIVISION BY WEIGHTS

(N = 11)

I = 11

0.57135446067443050D 02
-0.22931949503715550D 04
0.34449271098960500D 05
-0.26691108244368290D 06
0.12171732421673420D 07
-0.34859840680922630D 07
0.64479931659575390D 07
-0.76941942262113810D 07
0.57193040855838490D 07
-0.24083425386942900D 07
0.43874883894083490D 06

N = 12

I = 1

-0.62809712046008180D 00
0.97349208725481830D 02
-0.36740962081804510D 04
0.59164236204063700D 05
-0.50613790704194580D 06
0.25673908975000990D 07
-0.81825535131982310D 07
0.16806961478159360D 08
-0.22201731817903030D 08
0.18207319461823990D 08
-0.84295810775423920D 07
0.16828132146724960D 07

I = 2

0.97044216519729790D 00
-0.15036966861871660D 03
0.56705340420321710D 04
-0.91185184277295830D 05
0.77849448893553980D 06
-0.39383401739560010D 07
0.12509469255818680D 08
-0.25588333143220000D 08
0.33635220804863140D 08
-0.27424315890348060D 08
0.12611867398301090D 08
-0.24984179627599490D 07

(N = 12)

I = 3

-0.12744705976447860D 01
0.19737725411053610D 03
-0.74314329933348770D 04
0.11917694554194690D 06
-0.10134999401397680D 07
0.51007872028815800D 07
-0.16097249175721700D 08
0.32670693022309940D 08
-0.42551737919709900D 08
0.34329427598729830D 08
-0.15600219960168470D 08
0.30498673596812220D 07

I = 4

0.15997929592334350D 01
-0.24755198318340960D 03
0.92964442261903590D 04
-0.14842587620106170D 06
0.12542388006541950D 07
-0.62600943174096300D 07
0.19553828680241450D 08
-0.39205841039798720D 08
0.50356479548145220D 08
-0.40001461744518260D 08
0.17875659251661460D 08
-0.34334399481679000D 07

I = 5

-0.19896588693697860D 01
0.30747756727537110D 03
-0.11500307185105370D 05
0.18234946701209440D 06
-0.15258580461689860D 07
0.75200497695791070D 07
-0.23133310008012750D 08
0.45577110459247880D 08
-0.57424066378613350D 08
0.44696839423953880D 08
-0.19561620276687970D 08
0.36797047140185570D 07

ELEMENTS OF THE EXPLICIT INVERSE

MATRIX WITH DIVISION BY WEIGHTS

(N = 12)

I = 6

0.24981120367529350D	01
-0.38526531093196790D	03
0.14318886859062150D	05
-0.22461049637321410D	06
0.18513244138972080D	07
-0.89524265939160230D	07
0.26936331614258800D	08
-0.51812462789465410D	08
0.63678212531130240D	08
-0.48357186859487330D	08
0.20665514635039380D	08
-0.38006359881257360D	07

I = 7

-0.32133818690451120D	01
0.49394074202282790D	03
-0.18170262969558410D	05
0.28011641572128410D	06
-0.22544988126009910D	07
0.10593935323414500D	08
-0.30898406193109330D	08
0.57593609358735120D	08
-0.68693248668230210D	08
0.50737019856009050D	08
-0.21141481234343690D	08
0.38006359881257360D	07

I = 8

0.43050516901156380D	01
-0.65796811364996110D	03
0.23774517868851740D	05
-0.35571991162658850D	06
0.27530251921155250D	07
-0.12388827732748120D	08
0.34650435989768490D	08
-0.62178921116244410D	08
0.71725853799073080D	08
-0.51464395928094200D	08
0.20915131577516010D	08
-0.36797047140185370D	07

(N = 12)

I = 9

-0.61533573164492020D	01
0.93010244277608930D	03
-0.32449465759824320D	05
0.45[illegible]66972753631060D	06
-0.33205904613610080D	07
0.14064233019780170D	08
-0.373[illegible]9230994475820D	08
0.64362700153322790D	08
-0.71769601275455330D	08
0.50084066377137580D	08
-0.19892150178185280D	08
0.34334399481678820D	07

I = 10

0.96031948403621500D	01
-0.14440692792927150D	04
0.46326001574612040D	05
-0.57863539694709920D	06
0.38022143389776140D	07
-0.14995689763531230D	08
0.37813387059219770D	08
-0.62546016693489820D	08
0.67631326608678030D	08
-0.46069932779511790D	08
0.17946320996324770D	08
-0.30498673596811890D	07

I = 11

-0.19271827278777250D	02
0.26044176581899010D	04
-0.61421539686867610D	05
0.64792964056350660D	06
-0.38469474672538560D	07
0.14206472181404030D	08
-0.34242160151655240D	08
0.54835778671493580D	08
-0.57888553140108210D	08
0.38718629859131920D	08
-0.14870730192057450D	08
0.24984179627597960D	07

ELEMENTS OF THE EXPLICIT INVERSE
MATRIX WITH DIVISION BY WEIGHTS

(N = 12)

I = 12

0.674975778702486550D 02
-0.32085918288890160D 04
0.57375030486011780D 05
-0.53340516117338810D 06
0.29535236126991310D 07
-0.10448804284924580D 08
0.24480199520638750D 08
-0.38435239097431290D 08
0.39997175270061230D 08
-0.26466235493383090D 08
0.10081364283853400D 08
-0.16828132146722190D 07

N = 13

I = 1

0.62754331553197030D 00
-0.11358033610843180D 08
0.50250939417970050D 04
-0.95442184513324400D 05
0.97168744238521400D 06
-0.59404968247014980D 07
0.23229631009714890D 08
-0.60063715334341270D 08
0.10380582544494940D 09
-0.11855578799997500D 09
0.85833117642566710D 08
-0.35664862301875860D 08
0.64752096878575660D 07

I = 2

-0.96588072089159760D 00
0.17478290345705200D 03
-0.77282694742583250D 04
0.14663509169035800D 06
-0.14907030198521330D 07
0.90960216544733470D 07
-0.35483079564821880D 08
0.91477928348195400D 08
-0.15754736564692130D 09
0.17920322086746790D 09
-0.12913513896815610D 09
0.53371995158417440D 08
-0.96318464320360660D 07

(N = 13)

I = 3

0.12594286332923030D 01
-0.22781786911458130D 03
0.10061811146213320D 05
-0.19054009023605280D 06
0.19316439162564420D 07
-0.11743348528528470D 08
0.45600386924790600D 08
-0.11691192072352450D 09
0.20004546717896550D 09
-0.22584442129174520D 09
0.16137022920607280D 09
-0.66066587924320450D 08
0.11799267514586200D 08

I = 4

-0.15635930378826840D 01
0.28266984060442140D 03
-0.12461600812048860D 05
0.23524972654785430D 06
-0.23742867586914440D 07
0.14350435738888250D 08
-0.55323068110608670D 08
0.14062525506500070D 09
-0.23824303520694750D 09
0.26597547632329040D 09
-0.18771579270607220D 09
0.75836132834923010D 08
-0.13354193591861380D 08

I = 5

0.19142248647590760D 01
-0.34574583103880170D 03
0.15200112938246720D 05
-0.28559473259563270D 06
0.28631024340339590D 07
-0.17154969863530380D 08
0.65436802612953100D 08
-0.16428685625950120D 09
0.27448106194385310D 09
-0.30181562617390290D 09
0.20960921505548820D 09
-0.83281068929590510D 08
0.14419082659029250D 08

ELEMENTS OF THE EXPLICIT INVERSE
MATRIX WITH DIVISION BY WEIGHTS

(N = 13)

I = 6

-0.23511020637611940D 01
0.42407906939194600D 03
-0.18566223285048000D 05
0.34637477026633250D 06
-0.34378691560332090D 07
0.20337691079489820D 08
-0.76406714822997720D 08
0.18856146846996060D 09
-0.30924414061381730D 09
0.33353320311084930D 09
-0.22715895046998290D 09
0.88531198296311130D 08
-0.15044119928021750D 08

I = 7

0.29326171875000020D 01
-0.52787109375000080D 03
0.22962392578125020D 05
-0.42376318359375070D 06
0.41429083007812560D 07
-0.24052200046874980D 08
0.88433892421875160D 08
-0.21324162234375030D 09
0.34154465071289120D 09
-0.35991154417968810D 09
0.23973768959765660D 09
-0.91502934960937660D 08
0.15250489160156280D 08

I = 8

-0.37592935460218400D 01
0.67442121115956970D 03
-0.29035827449899610D 05
0.52662580414[illegible]8[illegible]910D 06
-0.50285159398617100D 07
0.28384930929298120D 08
-0.10125611070161880D 09
0.23691794605853040D 09
-0.36878972924484720D 09
0.37854677945665700D 09
-0.24622768445999690D 09
0.91998240839949890D 08
-0.15044119928021720D 08

(N = 13)

I = 9

0.5[illegible]275660326371340D 01
-0.89670538759823900D 03
0.37923801623981460D 05
-0.66768805526947810D 06
0.61340924932227870D 07
-0.33194249772054250D 08
0.11369963774122740D 09
-0.25646562287570360D 09
0.38662464671274010D 09
-0.38801591823593430D 09
0.24517691232592170D 09
-0.89747922978760040D 08
0.14419082659029160D 08

I = 10

-0.71800957850265000D 01
0.12666259857963150D 04
-0.51721949968460710D 05
0.38073246503961200D 06
-0.74051109220683070D 07
0.37765288364222390D 08
-0.12310923471440630D 09
0.26669449291041760D 09
-0.38903833027967030D 09
0.37811773502616730D 09
-0.23489510858476790D 09
0.84414190267412770D 08
-0.1335[illegible]193591861230D 08

I = 11

0.11435022541389190D 02
-0.19659144259814680D 04
0.73837337089061730D 05
-0.10871874022740590D 07
0.85006375595485470D 07
-0.40411727534986120D 08
0.12506633008148290D 09
-0.26052601146064870D 09
0.36875640203382980D 09
-0.35003438814031490D 09
0.21338941800123120D 09
-0.75524622250711790D 08
0.11799267514585840D 08

ELEMENTS OF THE EXPLICIT INVERSE
MATRIX WITH DIVISION BY WEIGHTS

(N = 13)

I = 12

-0.22477370421262190D 02
0.35453247719126610D 04
-0.98039763299629560D 05
0.12204159688198080D 07
-0.86285698775474890D 07
0.38431084088930520D 08
-0.11373773829282280D 09
0.22946582918523830D 09
-0.31718441412569740D 09
0.29569465014905760D 09
-0.17774505673993810D 09
0.62210162026013170D 08
-0.96318464320357350D 07

I = 13

0.78723215761735440D 02
-0.43733370845259480D 04
0.91749242665078710D 05
-0.10069243767794210D 07
0.66410657217231400D 07
-0.28341539376072510D 08
0.81542617182726970D 08
-0.16131015049826300D 09
0.21982054304058640D 09
-0.20275409315112290D 09
0.12088347172049520D 09
-0.42037653952402330D 08
0.64752096878556300D 07

N = 14

I = 1

-0.62710058676849130D 00
0.13105968903823980D C3
-0.67159735263579190D 04
0.14845758311417680D 06
-0.17713639267792060D 07
0.12814838745636220D 08
-0.60088842697869890D 08
0.18975509790027530D 09
-0.41111999022164400D 09
0.61198659250115030D 09
-0.61492202573029410D 09
0.39829832256181480D 09
-0.15007902306178370D 09
0.24984612699678530D 08

I = 2

0.96225356315207790D 00
-0.20107528446341820D 03
0.10299271598740270D 05
-0.22749551893141230D 06
0.27115024074627710D 07
-0.19588419447538950D 08
0.91687380045722230D 08
-0.28891943525264980D 09
0.62438119257867880D 09
-0.92670839285596560D 09
0.92801382186042630D 09
-0.59879739664643370D 09
0.22465816840735830D 09
-0.37221050666135830D 08

ELEMENTS OF THE EXPLICIT INVERSE
MATRIX WITH DIVISION BY WEIGHTS

(N = 14)

I = 3

-0.12475865212784120D 01
0.26062760717767190D 03
-0.13338370066251030D 05
0.29420284436129140D 06
-0.34994068475626250D 07
0.25212540726810810D 08
-0.11761754061302880D 09
0.36913741442143300D 09
-0.79397229603140680D 09
0.11720111890433050D 10
-0.11664350161564310D 10
0.74745627373425850D 09
-0.27839900529629120D 09
0.45724736356779090D 08

I = 4

0.15355288284604530D 01
-0.32064094843427940D 03
0.16387909462115830D 05
-0.36064900873599210D 06
0.42759286645134930D 07
-0.30677579805035340D 08
0.14236676990651250D 09
-0.44403634576476030D 09
0.94819133467043650D 09
-0.13882270207062360D 10
0.13690686886524820D 10
-0.86858664432220080D 09
0.31993822789747540D 09
-0.51968787273840710D 08

(N = 14)

I = 5

-0.18570160912215590D 01
0.38752227419995950D 03
-0.19767119057866770D 05
0.43355769719039980D 06
-0.51158922177818850D 07
0.36477408002075480D 08
-0.16800267212292770D 09
0.51933559363838030D 09
-0.10977565123736930D 10
0.15891593037308090D 10
-0.15481701995747850D 10
0.96954303861109500D 09
-0.35232510694594320D 09
0.56440868814093620D 08

I = 6

0.22429767477604370D 01
-0.46762438068047850D 03
0.23784307536616040D 05
-0.51912052591397640D 06
0.60832968396820860D 07
-0.42991511298815780D 08
0.19589166431292110D 09
-0.59810609757789330D 09
0.12470305777461890D 10
-0.17788305398299360D 10
0.17064392698708080D 10
-0.10519615804934060D 10
0.37628797956669640D 09
-0.59347261881679930D 08

ELEMENTS OF THE EXPLICIT INVERSE
MATRIX WITH DIVISION BY WEIGHTS

(N = 14)

I = 7

-0.27347513435818940D 01
0.56936165211156270D 03
-0.28835806993648890D 05
0.62485803994337690D 06
-0.72488607759361210D 07
0.50579039189955630D 08
-0.22702150561159070D 09
0.68159952264165800D 09
-0.13958347470868530D 10
0.19546601596146270D 10
-0.18408185547487770D 10
0.11145041508177720D 10
-0.39179754849029220D 09
0.60781758986943110D 08

I = 8

0.33973558713789260D 01
-0.70582687381741650D 03
0.35516457151659940D 05
-0.76127530737318370D 06
0.86993044124879000D 07
-0.59581548577651650D 08
0.26184043920873030D 09
-0.76867050936797620D 09
0.15387939241834260D 10
-0.21076999149043930D 10
0.19436719741531690D 10
-0.11539107699158290D 10
0.39836531833996820D 09
-0.60781758986943040D 08

(N = 14)

I = 9

-0.43454135994380020D 01
0.89977288895651130D 03
-0.44808975591674030D 05
0.94395747735251700D 06
-0.10537536408041470D 08
0.70205644801013560D 08
-0.29957197201463970D 09
0.85419165997620520D 09
-0.16637140741651880D 10
0.22222615089891290D 10
-0.20034483623153590D 10
0.11655922524640850D 10
-0.39522642489514270D 09
0.59347261881679870D 08

I = 10

0.58047100121171420D 01
-0.11950398764140700D 04
0.58456906195874110D 05
-0.11955987721625510D 07
0.12847342006460540D 08
-0.82123426856577030D 08
0.33681964482398110D 09
-0.92684340544247790D 09
0.17500898366654430D 10
-0.22760221055625940D 10
0.20054346475457740D 10
-0.11440295227590730D 10
0.38140618763727160D 09
-0.56440868814093220D 08

ELEMENTS OF THE EXPLICIT INVERSE MATRIX WITH DIVISION BY WEIGHTS

(N = 14)

I = 11

-0.82853474326718680D 01
0.16869318547414000D 04
-0.79686780156054520D 05
0.15411315559997220D 07
-0.15521806435712500D 08
0.93602951931118650D 08
-0.36572146347742750D 09
0.96739086896435690D 09
-0.17689635876697900D 10
0.22410783352539520D 10
-0.19325345179767670D 10
0.10828933169120550D 10
-0.35565600666244880D 09
0.51968787273839930D 08

I = 12

0.13192182617824440D 02
-0.26176698889117130D 04
0.11376129841733790D 06
-0.19486373264239120D 07
0.17854602486996220D 08
-0.10045213191534110D 09
0.37287171793200110D 09
-0.94896563883296660D 09
0.16845659139820480D 10
-0.20851614127761090D 10
0.17651242434039460D 10
-0.97439764600749770D 09
0.31612256734182530D 09
-0.45724736356777360D 08

(N = 14)

I = 13

-0.25929439201462620D 02
0.47205431754915600D 04
-0.15122645500080860D 06
0.21917930943188830D 07
-0.18170314609712520D 08
0.95819965435587990D 08
-0.34025026907390060D 09
0.83890787197131280D 09
-0.14546416517418060D 10
0.17686811664737440D 10
-0.14765389168794350D 10
0.80614132771666560D 09
-0.25921549025238640D 09
0.37221050666132720D 08

I = 14

0.90812360683100790D 02
-0.58289663908486240D 04
0.14173147839979980D 06
-0.18115731489670740D 07
0.14012662197212460D 08
-0.70814860274995020D 08
0.24448659025150680D 09
-0.59112499164761420D 09
0.10105715360278980D 10
-0.12157870827753120D 10
0.10067432324796280D 10
-0.54614983639513160D 09
0.17472094203397150D 09
-0.24984612699669120D 08

ELEMENTS OF THE EXPLICIT INVERSE MATRIX WITH DIVISION BY WEIGHTS

N = 15

I = 1

0.62674114724037670D	00
-0.14978732641334850D	03
0.87991702158178660D	04
-0.22384410147813690D	06
0.30907278460131640D	07
-0.26070735053359080D	08
0.14398922790202560D	09
-0.54295844801453790D	09
0.14312370236968300D	10
-0.26615478102941000D	10
0.34745206974963210D	10
-0.31142349585236980D	10
0.18242613812678980D	10
-0.62870665172053600D	09
0.96634843254008150D	08

I = 2

-0.95932021867915070D	00
0.22924661107392350D	03
-0.13462431736329460D	05
0.34227829881890100D	06
-0.47221560871994830D	07
0.39789495277815740D	08
-0.21946537164326570D	09
0.82623583095249640D	09
-0.21738324227372380D	10
0.40336319906660950D	10
-0.52525395858817300D	10
0.46945865520285570D	10
-0.27413061611607320D	10
0.94143312509001210D	09
-0.14414037028721530D	09

(N = 15)

I = 3

0.12380871459534810D	01
-0.29580129552387820D	03
0.17359797991491830D	05
-0.44089167033424270D	06
0.60732983614250210D	07
-0.51071425638271780D	08
0.28098796648939410D	09
-0.10546746851127840D	10
0.27650829106754510D	10
-0.51099381550103200D	10
0.66235585027415350D	10
-0.58895796177453840D	10
0.34195573387889630D	10
-0.11670455028865680D	10
0.17747321082681620D	09

I = 4

-0.15132903440408250D	01
0.36143458277718120D	03
-0.21190591940268440D	05
0.53727739053594980D	06
-0.73832858170019010D	07
0.61893331012305830D	08
-0.33921194402664700D	09
0.12673395948070550D	10
-0.33048142182154370D	10
0.60700974692866130D	10
-0.78144193438138940D	10
0.68962211361328180D	10
-0.39712890466219770D	10
0.13434429202464070D	10
-0.20239306917933810D	09

ELEMENTS OF THE EXPLICIT INVERSE MATRIX WITH DIVISION BY WEIGHTS

(N = 15)

I = 5

```
 0.18124564883614930D 01
-0.43268210651761890D 03
 0.24331031296391310D 05
-0.64067995698336990D 06
 0.87735879029932980D 07
-0.73215645062549870D 08
 0.39903681011349510D 09
-0.14810688760755490D 10
 0.38330647941031090D 10
-0.69809556413282740D 10
 0.89037946618235820D 10
-0.77791485387328610D 10
 0.44322964323493450D 10
-0.14627963179240750D 10
 0.22063449928279640D 09
```

I = 6

```
-0.21607164234822870D 01
 0.51547223690100050D 03
-0.30115551313185460D 05
 0.75902806615901360D 06
-0.10342913718251820D 08
 0.85762161473356650D 08
-0.46380045512426430D 09
 0.17059278828690470D 10
-0.43701651364934290D 10
 0.78705684074366690D 10
-0.99189402533916700D 10
 0.85580055666431470D 10
-0.48135128963171870D 10
 0.15894035471711490D 10
-0.23363534134708420D 09
```

(N = 15)

I = 7

```
 0.25889505065780910D 01
-0.61703749015220440D 03
 0.35942839536095310D 05
-0.90138035701719990D 06
 0.12196656755473700D 08
-0.10023012892975070D 09
 0.53625232965057010D 09
-0.19483799634265430D 10
 0.49245086201087030D 10
-0.87428855747374130D 10
 0.10856370799125670D 11
-0.92276471019543550D 10
 0.51135852446701240D 10
-0.16640995423502350D 10
 0.24119471694668040D 09
```

I = 8

```
-0.31420898437500060D 01
 0.74781738281250160D 03
-0.43373408203125070D 05
 0.10798483007812500D 07
-0.14464283818359400D 08
 0.11736246005859380D 09
-0.61863940810547000D 09
 0.22110096937500030D 10
-0.54918100786377040D 10
 0.95783965132324390D 10
-0.11686231979296880D 11
 0.97646703451172080D 10
-0.53235100375488370D 10
 0.17058761446289080D 10
-0.24369659208984420D 09
```

ELEMENTS OF THE EXPLICIT INVERSE MATRIX WITH DIVISION BY WEIGHTS

(N = 15)

I = 9

0.38931009622298320D 01
-0.92459692959775470D 03
0.53278534088889670D 05
-0.13120350817553940D 07
0.17312394046906410D 08
-0.13791130423589840D 09
0.71203877735789710D 09
-0.24897282238044170D 10
0.60497726835068710D 10
-0.10329636183294670D 11
0.12352321377315810D 11
-0.10130488499360400D 11
0.54290104362649800D 10
-0.17126264949032910D 10
0.24119471694668040D 09

I = 10

-0.49721421265370650D 01
0.11769003011083430D 04
-0.67117059857607340D 05
0.16244931453902490D 07
-0.20943608396814680D 08
0.16235745200514600D 09
-0.81442361197469660D 09
0.27681423968320320D 10
-0.65498909091008840D 10
0.10915782300378510D 11
-0.12772292374733760D 11
0.10273936760151980D 11
-0.54120828456768640D 10
0.16814912316880170D 10
-0.23363534134708260D 09

(N = 15)

I = 11

0.66366757805783670D 01
-0.15618639807458830D 04
0.87490785271139800D 05
-0.20562089908785320D 07
0.25527491884002960D 08
-0.18999218192076590D 09
0.91677654852783980D 09
-0.30096847588033260D 10
0.69095899441064070D 10
-0.11220336871860480D 11
0.12840312126592360D 11
-0.10134053583039180D 11
0.52518837322508010D 10
-0.16088866717550650D 10
0.22083449926279530D 09

I = 12

-0.94692047715028570D 01
0.22038877080534270D 04
-0.11922583550614790D 06
0.26504614225451120D 07
-0.30863745176710530D 08
0.21688274440203220D 09
-0.99775745931098660D 09
0.31508987956921740D 10
-0.70097464706489320D 10
0.11094760259234770D 11
-0.12431850981447720D 11
0.96417768253899200D 10
-0.49243003787383290D 10
0.14900600482642920D 10
-0.20239306917933360D 09

ELEMENTS OF THE EXPLICIT INVERSE
MATRIX WITH DIVISION BY WEIGHTS

(N = 15)

I = 13

0.15074711637672710D 02
-0.34193093774842380D 04
0.17021473689403690D 06
-0.33542232418030070D 07
0.35561798593828060D 08
-0.23330092608529340D 09
0.10202340266802310D 10
-0.31013504180674590D 10
0.67005078094155080D 10
-0.10365215941370120D 11
0.11404637279697670D 11
-0.87158079635305240D 10
0.43980279865036230D 10
-0.13175794486687880D 10
0.17747321082680670D 09

I = 14

-0.29628043653750330D 02
0.61660581986535680D 04
-0.21648742165995170D 06
0.37788653936583870D 07
-0.36266797135704260D 08
0.22309270240057440D 09
-0.93354340420152000D 09
0.27498478339129390D 10
-0.58042945429329880D 10
0.88211958375875230D 10
-0.95729310319341260D 10
0.72363984094249240D 10
-0.36194492311267830D 10
0.10765320589308860D 10
-0.14414037028719950D 09

(N = 15)

I = 14

0.10376501328106770D 03
-0.76202104731309220D 04
0.21252068208342310D 06
-0.31278310698445420D 07
0.28013358416827970D 08
-0.16516077951213570D 09
0.67203255776888300D 09
-0.19413456172002270D 10
0.40403272170379310D 10
-0.60759405391412580D 10
0.65405630226025860D 10
-0.49128657269458650D 10
0.24448456450144930D 10
-0.72418115383522920D 09
0.96634843253961540D 08

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13 ABSTRACT

The direct problem in transient heat conduction requires the determination of conditions at an interior location when conditions are known at the boundaries of a solid. Conversely, the inverse problem requires the determination of conditions at the boundaries of a solid when conditions are known at an interior location. Consequently special methods are required in the solution of the inverse problem. A new method, numerical inversion of the Laplace transform, is used to solve this complex problem. Application of this numerical technique to the semi-infinite solid, "long" cylinder, and sphere is made, and the accuracy of solution is discussed. This method of solution provides the engineer with a simple, powerful tool that can be used in the determination of heat transfer phenomena in a solid.

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